Two Gauss-Bonnet and Poincaré-Hopf Theorems for Orbifolds with Boundary

by

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Thesis directed by Assoc. Prof. Carla Farsi

The goal of this work is to generalize the Gauss-Bonnet and Poincaré-Hopf Theorems to the case of orbifolds with boundary. We present two such generalizations, the first in the spirit of [20], which uses an argument parallel to that contained in [22]. In this case, the local data (i.e. integral of the curvature in the case of the Gauss-Bonnet Theorem and the index of the vector field in the case of the Poincaré-Hopf Theorem) is related to Satake's orbifold Euler characteristic, a rational number which depends on the orbifold structure.

For the second pair of generalizations, we use a more recent orbifold cohomology [3] to express the local data in a way which can be related to the Euler characteristic of the underlying space of the orbifold. This case applies only to orbifolds which admit almost-complex structures.

Dedication

To Michael Thomas Seaton and Christina Heather Bost, my siblings old and new, with congratulations and best wishes.

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Chapter 1

Introduction

An orbifold is perhaps the simplest case of a singular space; it is a topological space which is locally diffeomorphic to $\mathbb{R}^n/G$ where G is a finite group. Orbifolds were originally introduced by Satake in [19] and [20], where they were given the name V-manifold, and rediscovered by Thurston in [24], where the term orbifold was coined. Satake and Thurston's definitions differ, however, in that Satake required the group action to have a fixed point set of codimension at least two, while Thurston did not. Hence, Thurston's definition allows group actions such as reflections through hyperplanes. Today, authors differ on whether or not this requirement is made; often, when it is, the orbifolds are referred to as **codimension-2 orbifolds**. It is these orbifolds which are our object of study.

The point of view of this work is that an orbifold structure is a generalization of a differentiable structure on a manifold. We do not mean to suggest that the underlying space of an orbifold is necessarily a topological manifold; this is only the case in dimension ≤ 2, and not even in dimension 1 if the codimension-2 requirement is lifted. However, there are many examples of orbifolds whose underlying topological spaces are indeed manifolds. In these cases, we view the orbifold structure as a singular differentiable structure on the manifold. It should be noted that this is used as a guiding principle only, and that our results apply in general.

Hence, we improve upon Satake's Gauss-Bonnet theorem for orbifolds [20] by de-

veloping a Gauss-Bonnet integrand (and corresponding **orbifold Euler Class**) whose integral relates to the Euler Characteristic of the underlying topological space, as opposed to the orbifold Euler Characteristic (Theorem 4.4.2). This result depends on recent developments in the theory of orbifolds, most notably the Orbifold Cohomology Theory of Chen-Ruan [3], and hence is restricted to the case of an orbifold which admits an almost complex structure.

As is well-known, the Gauss-Bonnet theorem is very closely related to the Poincaré-Hopf theorem; indeed, either of the two theorems can be viewed as a corollary of the other. With a new Gauss-Bonnet theorem, then, comes a new Poincaré-Hopf theorem. Following the work of Sha on the secondary Chern-Euler class for manifolds [22], we generalize the Poincaré-Hopf theorem to the case of orbifolds with boundary (Corollary 4.4.4).

To this end, a remark is in order. Satake's original definition of an orbifold with boundary was in many senses not very strict. In particular, the boundaries of his orbifolds were not necessarily orbifolds. This allows little control over vector fields on the boundary; indeed, it is not generally the case that the orbifold is locally a product near the boundary, and hence there need not exist a vector field which does not vanish on the boundary. Since then, a more natural definition of orbifold with boundary has been given in [24]. Using this definition, we are able to refine Satake's original Gauss-Bonnet theorem for the case with boundary using his original techniques.

The outline of this work is roughly as follows. In Chapter 2, we collect the necessary background information on orbifolds and orbifolds with boundary, including several examples, paying particular attention to the behavior of vector fields on orbifolds. In Chapter 3, we review Satake's Gauss-Bonnet and Poincaré-Hopf theorems for orbifolds and orbifolds with boundary, making improvements where possible using the modern definition of an orbifold with boundary (see Theorem 3.2.2). We also apply the arguments of Sha [22] to characterize the boundary term in the case with boundary as the

evaluation of a secondary characteristic class on the boundary (see Theorem 3.4.2). It is easy to see that, even in the case of a manifold, the boundary term of this formula will always depend on the vector field. For instance, consider a fixed vector field on S^2 with one singular point p. Remove an open disk to produce a vector field on a manifold with boundary. The index of the vector field depends on whether p is contained in the disk removed, but the Euler Characteristic does not; hence, the boundary term must depend on the vector field. The spirit of the Poincaré-Hopf theorem is that this term should be formulated in a manner as independent of the vector field as possible; it is this reason that we chose the result of Sha to generalize to orbifolds.

In Chapter 4, we review the Chen-Ruan orbifold cohomology, and extend it in a straightforward manner to the case with boundary. Loosely speaking, the idea of this cohomology theory is to associate to an orbifold Q another orbifold, $\tilde{Q}$ (where at least one of the connected components of $\tilde{Q}$ is diffeomorphic to Q), and use the cohomology groups of $\tilde{Q}$ (we should note that this informal description leaves out an important modification to the grading of the groups). Using this cohomology theory, we develop an Euler Class which relates to the Euler Characteristic of the underlying topological space of Q. The essential idea here is to apply the Chern-Weil description of characteristic classes to the curvature of a connection on $\tilde{Q}$, yielding a (non-homogeneous) characteristic class in orbifold cohomology. Similarly, the index of a vector field X on Q is computed to be the index of its pull-back $\tilde{X}$ onto $\tilde{Q}$. This suggests the paradigm that geometric structures on $\tilde{Q}$ can be considered to be structures on Q which take multiple values on singular sets. It is in this manner that we prove the aforementioned Gauss-Bonnet and Poincaré-Hopf theorems for orbifolds (Theorem 4.4.2 and Corollary 4.4.4) and orbifolds with boundary (Theorems 4.5.1 and 4.5.2), relating to the Euler Characteristics of the underlying space.

Theorems, definitions, examples, etc. are numbered sequentially according to the section in which they appear. So Theorem $X.Y.Z$ is in Chapter X, Section Y,

and follows Definition (or Lemma, Example, etc.) $X.Y.Z-1$. Figures are numbered independently according to the Chapter in which they appear.

Chapter 2

Orbifolds and Their Structure

2.1 Definitions and Examples

In this section, we collect the definitions and background we will need. For more information, the reader is referred to the original work of Satake in [19] and [20]. As well, [17] contains as an appendix a thorough introduction to orbifolds, focusing on their differential geometry. Other good introductions include [24] and [2], the former providing a great deal of information on the topology of low-dimensional orbifolds. The reader is warned that the definition used in these latter two works is more general than ours, as it admits group actions which fix sets of codimension 1. For the most part, we follow the spirit of Satake and Ruan.

2.1.1 Orbifolds and Orbifolds With Boundary

Let X_Q be a Hausdorff space.

Definition 2.1.1 (orbifold chart) Let $U \subset X_Q$ be a connected open set. A ($\mathcal{C}^\infty$) **orbifold chart** for U (also known as a ($\mathcal{C}^\infty$) **local unifomizing system**) is a triple $\{V, G, \pi\}$ where

- V is an open subset of $\mathbb{R}^n$,
- G is a finite group with a ($\mathcal{C}^\infty$) action on V such that the fixed point set of any $\gamma \in G$ which does not act trivially on V has codimension at least 2 in V, and

- $\pi : V \to U$ is a surjective continuous map such that $\forall \gamma \in G$, $\pi \circ \gamma = \pi$ that induces a homeomorphism $\tilde{\pi} : V/G \to U$.

If G acts effectively on V, then the chart is said to be **reduced**.

The definition of the appropriate notion of 'chart' for orbifolds with boundary is similar:

Definition 2.1.2 (orbifold chart with boundary) Let $U \subset X_Q$ be a connected open set. A ($\mathcal{C}^\infty$) **orbifold chart with boundary** or ($\mathcal{C}^\infty$) **local unifomizing system** for U is a triple $\{V, G, \pi\}$ where

- V is an open subset of $\mathbb{R}^n_+ := \{(x_1, x_2, \ldots x_n) \in \mathbb{R}^n : x_1 \geq 0\}$,
- G is a finite group with a ($\mathcal{C}^\infty$) action on V such that the fixed point set of any $\gamma \in G$ which does not act trivially on V has codimension at least 2 in V, and such that $\gamma \partial \mathbb{R}^n_+ \subset \partial \mathbb{R}^n_+$, and
- $\pi : V \to U$ is a surjective continuous map such that $\forall \gamma \in G$, $\pi \circ \gamma = \pi$ that induces a homeomorphism $\tilde{\pi} : V/G \to U$.

Again, if G acts effectively on V, then the chart is said to be **reduced**.

If $V \cap \partial \mathbb{R}^n_+ = \emptyset$, then $\{V, G, \pi\}$ is an ordinary orbifold chart; for emphasis, we may refer to these as **orbifold charts without boundary**. Note also that if $\{V, G, \pi\}$ is an orbifold chart with boundary for some set U, then restricting the chart to $\partial V = \partial \mathbb{R}^n_+ \cap V$, it is clear that $\{\partial V, G, \pi_{|\partial V}\}$ is an orbifold chart without boundary for $\pi(\partial V)$.

We will always use the notation that if V_i is the domain of a chart, then G_i is the group of the chart, π_i the projection for the chart, and U_i the range of the chart; i.e. items with the same subscript correspond to the same chart.

Orbifold charts relate to one another via **injections**.

Definition 2.1.3 (injection) If $\{V_i, G_i, \pi_i\}$ and $\{V_j, G_j, \pi_j\}$ are two orbifold charts (with or without boundary) for U_i and U_j, respectively, where $U_i \subset U_j \subset X_Q$, then an **injection** $\lambda_{ij} : \{V_i, G_i, \pi_i\} \to \{V_j, G_j, \pi_j\}$ is a pair $\{f_{ij}, \phi_{ij}\}$ where

- $f_{ij} : G_i \to G_j$ is an injective homomorphism such that if K_i and K_j denote the kernel of the action of G_i and G_j, respectively, then f_{ij} restricts to an isomorphism of K_i onto K_j, and

- $\phi_{ij} : V_i \to V_j$ is a smooth embedding such that $\pi_i = \pi_j \circ \phi_{ij}$ and such that for each $\gamma \in G_i$, $\phi_{ij} \circ \gamma = f_{ij}(\gamma) \circ \phi_{ij}$. If the charts have boundary then $\phi_{ij}(\partial V_i) \subseteq \partial V_j$.

Given an orbifold chart $\{V, G, \pi\}$ (with or without boundary), each $\gamma \in G$ induces an injection λ_γ of $\{V, G, \pi\}$ into itself via

$$\begin{aligned} f_\gamma &: G \to G \\ &: \gamma' \mapsto \gamma\gamma'\gamma^{-1} \\ \phi_\gamma &: V \to V \\ &: x \mapsto \gamma x. \end{aligned}$$

Note that this injection is trivial if γ acts trivially. Similarly, given an injection $\lambda_{ij} : \{V_i, G_i, \pi_i\} \to \{V_j, G_j, \pi_j\}$, each element $\gamma \in G_j$ defines an injection $\gamma\lambda_{ij} : \{V_i, G_i, \pi_i\} \to \{V_j, G_j, \pi_j\}$ with $\gamma\lambda_{ij} := \{\gamma f_{ij}\gamma^{-1}, \gamma \circ \phi_{ij}\}$. Moreover, every two injections of $\{V_i, G_i, \pi_i\}$ into $\{V_j, G_j, \pi_j\}$ are related in this manner (see [20], Lemma 1).

Two orbifold charts $\{V_i, G_i, \pi_i\}$ and $\{V_j, G_j, \pi_j\}$ are said to be **equivalent** if $U_i = U_j$, and there is an injection λ_{ij} with f_{ij} an isomorphism and ϕ_{ij} a diffeomorphism.

Definition 2.1.4 (orbifold) An **orbifold** Q is a Hausdorff space X_Q, the **underlying space of** Q, together with a family $\mathcal{F}$ of orbifold charts (without boundary) such that

- Each $p \in X_Q$ is contained in an open set U_i covered by an orbifold chart $\{V_i, G_i, \pi_i\} \in \mathcal{F}$. If $p \in U_i \cap U_j$ for U_i and U_j uniformized sets, then there is a uniformized set U_k such that $p \in U_k \subset U_i \cap U_j$.

- Whenever $U_i \subset U_j$ for two uniformized sets, there is an injection $\lambda_{ij} : \{V_i, G_i, \pi_i\} \to \{V_j, G_j, \pi_j\}$.

If each chart in $\mathcal{F}$ is reduced, then Q is said to be a **reduced** orbifold. Otherwise, Q is **unreduced**.

The definition of an orbifold with boundary is identical, except that it allows orbifold charts with boundary. In this case, $\partial Q := \{\pi_i(\partial V_i) : \{V_i, G_i, \pi_i\} \in \mathcal{F}\}$ is the **boundary** of Q. It is easy to see that the restrictions $\{\partial V_i, G_i, (\pi_i)_{|\partial V_i}\}$ endow ∂Q with the structure of an orbifold. We will sometimes refer to an orbifold as an **orbifold without boundary** for emphasis.

It is easy to see that, given an unreduced orbifold Q, one can associate to it a reduced orbifold Q_{red} by redefining the group in each chart to be G_i/K_i, where K_i again denotes the kernel of the G_i-action on V_i.

Fix $p \in Q$, and say $p \in U$ for some set $U \subset Q$ uniformized by $\{V, G, \pi\}$. Let $\tilde{p} \in V$ such that $\pi(\tilde{p}) = p$, and let $I_{\tilde{p}}$ denote the isotropy subgroup of $\tilde{p} \in V$. The isomorphism class of $I_{\tilde{p}}$ depends only on p; indeed, if $\tilde{p}'$ is another choice of a lift, then there is a group element $\gamma \in G$ such that $\gamma\tilde{p}' = \tilde{p}$, so that $I_{\tilde{p}}$ and $I_{\tilde{p}'}$ are conjugate via γ. Similary, if $U', \{V', G', \pi'\}$ is another choice of chart with $p \in U'$ (and we assume, by shrinking domains if necessary, that $U \subseteq U'$), then there is an injection $\lambda : \{V, G, \pi\} \to \{V', G', \pi'\}$ with associated homomorphism $f : G \to G'$ that maps $I_{\tilde{p}}$ isomorphically onto $I_{\phi(\tilde{p})}$ (see [20], page 468). We will often refer to the (isomorphism class) of this group as the **isotropy group of** p, denoted I_p. If $I_p \neq 1$ (with respect to the orbifold structure of Q_{red} in the case that Q is not reduced), then p is **singular**; otherwise, it is **nonsingular**. The collection of singular points of Q is denoted Σ_Q.

2.1.2 Examples

Before proceeding, we give some examples of orbifolds.

Example 2.1.5 If M is a smooth manifold and G a group that acts properly discontinuously on M such that the fixed point set of each element of G has codimension ≤ 2, then the quotient M/G is an orbifold ([20], [24]).

Similarly, we have the following claim.

Claim 2.1.6 If M is a manifold with boundary and G a group with action as above that fixes the boundary, then M/G is an orbifold with boundary.

This is proven as follows, following Thurston's proof ([24]) for the case of manifolds without boundary.

Let $p \in Q$, and let $\tilde{p} \in M$ project to p. Note that, by hypothesis, if $\tilde{p} \in \partial M$, then any point projecting to p is also an element of ∂M. Hence, we may define ∂Q to be the set of points which are projections of points in ∂M.

Now, in the case that p is not an element of ∂Q, we may use Thurston's proof (by intersecting the neighborhood of $\tilde{p}$ with the interior of M) to produce a chart near p. If $p \in \partial Q$, then let $I_{\tilde{p}}$ denote the isotropy group of $\tilde{p}$. There is a neighborhood $V_{\tilde{p}}$ of $\tilde{p}$ invariant by $I_{\tilde{p}}$ and disjoint from its translates by elements of G not in $I_{\tilde{p}}$. Note that $V_{\tilde{p}}$ is diffeomorphic to a neighborhood of $\mathbb{R}^n_+$, where n is the dimension of M, and that $\partial\mathbb{R}^n_+$ is clearly G-invariant. The projection $U_p = V_{\tilde{p}}/I_{\tilde{p}}$ is by fiat a homeomorphism.

To obtain a suitable cover of M/G, augment some cover $\{U_p\}$ by adjoining finite intersections. Whenever $U_{p_1} \cap \cdots \cap U_{p_k} \neq \emptyset$, this means some set of translates $\gamma_1 V_{p_1} \cap \cdots \cap \gamma_k V_{p_k}$ has a corresponding non-empty intesection. This intersection may be taken to be $U_{p_1} \cap \cdots \cap U_{p_k}$ with associated group $\gamma_1 I_1 \gamma_1^{-1} \cap \cdots \cap \gamma_k I_k \gamma_k^{-1}$.

With this, we need only note that Thurston's proof (applied to the boundary, and restricted to ∂M) will give a cover of ∂Q, consistent with that on Q, making ∂Q an orbifold. This completes the proof.

Orbifolds which arise in this way are called **global quotient orbifolds** or **global quotients**. They are examples of **good orbifolds,** orbifolds whose universal cover is a manifold or, equivalently, orbifolds diffeomorphic to the quotient of a manifold by the proper action of a discrete group. Orbifolds which are not good are called **bad orbifolds**.

Example 2.1.7 (Kawasaki [9]) If G is a Lie group that acts smoothly on a smooth manifold M such that

- $\forall x \in M$, the isotropy subgroup G_x is compact,
- $\forall x \in M$, there is a smooth slice S_x at x,
- $\forall x, y \in M$ such that $y \notin Gx$, there are slices S_x and S_y with $GS_x \cap GS_y = \emptyset$, and
- the dimension of G_x is constant on M,

then M/G is an orbifold. Moreover, every n-dimensional orbifold Q can be expressed in this manner with M the orthonormal frame bundle of Q and $G = O(n)$.

Example 2.1.8 The simplest example of an orbifold is that of a point p with the trivial action of a finite group G. Note that these orbifolds are not reduced unless $G = 1$.

Example 2.1.9 The $\mathbb{Z}_k$-teardrop is a well-known example of a bad orbifold, whose underlying topological space is S^2. It has one singular point, covered by a chart $\{V_1, G_1, \pi_1\}$ where V_1 is an open disk in $\mathbb{R}^2$ about the origin on which $G_1 = \mathbb{Z}_k$ acts via rotations, and every other point is covered by a chart with the trivial group (see Figure 2.1)).

In fact, every compact 2-dimensional orbifold (with or without boundary) can be constructed from a compact 2-dimensional manifold by removing disks and replacing them with $\mathbb{R}^2/G$ for some finite cyclic group $G < SO(2)$ with one singular point. Note that in dimension 2, the boundary of the orbifold cannot contain singular points.

The underlying space of an orbifold need not be a topological manifold, even in dimensions 3, as is demonstrated by the following example (taken from [23]).

Example 2.1.10 Let $\mathbb{Z}_2$ act on $\mathbb{R}^3$ via the antipodal map, and then the origin is the only fixed point. Then $\mathbb{R}^3/\mathbb{Z}_2$ is clearly an orbifold with one singular point, yet its underlying space is homeomorphic to a cone on $\mathbb{RP}^2$, which is not a manifold.

Example 2.1.11 (The $\mathbb{Z}_k$-$\mathbb{Z}_l$-Solid Hollow Football) An example of a bad orbifold with boundary is the **$\mathbb{Z}_k$-$\mathbb{Z}_l$-solid hollow football** with $k \neq l$. It is homeomorphic to the manifold with boundary $\{(x, y, z) \in \mathbb{R}^3 : 1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2\}$ with two boundary components, both homeomorphic to S^2. Its orbifold structure, however, is such that both of the boundary components have the orbifold structure of the $\mathbb{Z}_k$-$\mathbb{Z}_l$-football (the sphere with two singular points having respective groups $\mathbb{Z}_k$ and $\mathbb{Z}_l$; see [2], page 16, Example 10), and the interior has two singular sets, both homeomorphic to a line segment, with isotropy subgroups of order k and l, respectively (see Figure 2.2).

Note that this orbifold is good when $k = l$. For in this case, the orbifold can be expressed as $M/\mathbb{Z}_k$ where $M = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2\}$ and $\mathbb{Z}_k$ acts via rotation about the z-axis. That this orbifold with boundary is bad whenever $k \neq l$ follows from the fact that the $\mathbb{Z}_k$-$\mathbb{Z}_l$-football is bad and the following proposition.

Proposition 2.1.12 Let Q be a good orbifold with boundary. The ∂Q is a good orbifold.

Proof:

If $Q = M/G$ for some manifold M and group G, then $\partial Q = \partial M/(G_{|\partial M})$.

Q.E.D.

2.1.3 Structures on Orbifolds

The next step is to introduce the appropriate notion of a vector bundle on an orbifold. The following definition follows [11] (compare [20] and [17]; note that our definition of an orbifold vector bundle corresponds to Ruan's definition of a **good** orbifold vector bundle).

Definition 2.1.13 (orbifold vector bundlle) Let Q be a connected orbifold. By an **orbifold vector bundle** E **of rank** l, we mean a collection consisting, for each set $U \subset Q$ uniformized by $\{V, G, \pi\}$, of a G-bundle E_V over V of rank l such that the G-action on V and E_V have the same kernel. We require that for each injection $\lambda_{12} : \{V_1, G_1, \pi_1\} \to \{V_2, G_2, \pi_2\}$, there is a bundle map $\phi_{12}^* : (E_{V_2})_{|\phi_{12}(V_1)} \to E_{V_1}$ such that if $\phi_{12} \circ \gamma_1 = \gamma_2 \circ \phi_{12}$ for some $\gamma_i \in G_i$, then $\gamma_1 \circ \phi_{12}^* = \phi_{12}^* \circ \gamma_2$. The **total space** of the bundle E, also denoted E, is formed from the collection E_V/G by identifying points $(p, v) \in E_{V_1}$ and $(q, w) \in E_{V_2}$ whenever there is an injection $\lambda_{12} : \{V_1, G_1, \pi_1\} \to \{V_2, G_2, \pi_2\}$ such that $q = \phi_{12}(p)$ and $\phi_{12}^*(w) = v$. It is clear that the total space of a bundle E is an orbifold. Moreover, if ρ_V denotes the projection $\rho_V : E_V \to V$ for each V, then the collection of these projections patch together to form a well-defined map $\rho : E \to Q$, called the **projection** of the bundle E. In the case where Q is not connected, we will require that the rank of the E_V is constant only on the connected components of Q.

By a **section** s of an orbifold bundle E, we mean a collection s_V of sections of the G-bundles $\{E_V : \{V, G, \pi\}$ is an orbifold chart for Q $\}$ such that if $\lambda_{ij} : \{V_i, G_i, \pi_i\} \to \{V_j, G_j, \pi_j\}$ is an injection, then $\phi_{ij}^* s_{V_j}[\phi_{ij}(\tilde{p})] = s_{V_i}(\tilde{p})$ for each $\tilde{p} \in V$. We require that if $\tilde{p} \in V$ is fixed by $\gamma \in G$, then $s(\tilde{p})$ is also fixed by γ.

It is clear that this collection defines a well-defined map $s : Q \to E$ from Q into the total space of the bundle E such that $\rho \circ s$ is the identity on Q.

It is important to notice that E is not generally a vector bundle over Q, as the fiber $\rho^{-1}(p)$ over a point $p \in Q$ is not always a vector space. In general, $\rho^{-1}(p) \cong \mathbb{R}^l / I_p$, which is a vector space only when $I_{\tilde{p}}$ acts trivially; i.e. when p is nonsingular.

In particular, the **tangent bundle** $\rho : TQ \to Q$ of an orbifold is defined to be the collection of tangent bundles TV over each V where the G-action is given by the Jacobian of the injection induced by the group action. More explicitly, each $\gamma \in G$ induces an injection $\lambda_\gamma : \{V, G, \pi\} \to \{V, G, \pi\}$, as was explained in Subsection 2.1.1. For $\tilde{p} \in V$ and $p \in U$ such that $\pi(\tilde{p}) = p$, let

$$g_\gamma(\tilde{p}) = \left(\frac{\partial (u^i \circ \gamma)}{\partial u^j} \right)$$

be the Jacobian matrix of the action of γ at $\tilde{p}$ for a fixed choice $\{u^i\}$ of coordinates for V. Then $\{g_\gamma(\tilde{p}) : \gamma \in I_{\tilde{p}}\}$ defines an $I_{\tilde{p}}$-action on $T_{\tilde{p}} V_i$ (see [20]). Note that the space of vector fields on a uniformized set U, i.e. of sections of the orbifold tangent bundle over U, corresponds to the space of G-invariant vector fields on V. If $p \in Q$, we refer to the maximal vector space in $T_p Q := \rho^{-1}(p)$ as the **space of tangent vectors at** p**.** The tangent vectors at a singular point are tangent to the singular set.

In particular, given a metric on TQ (i.e. a G-invariant metric on Q), the exponential map exp: $TQ \to Q$ is defined. Using this map and the fact that the G-action on the tangent bundle is linear, given an orbifold chart $\{V, G, \pi\}$, we define an equivalent orbifold chart $\{V', G', \pi'\}$ such that $U = U'$ (i.e. $\pi(V) = \pi'(V')$), and such that G' acts on $V' \subseteq \mathbb{R}^n$ as a subgroup of $O(n)$. Picking a point $p \in U'$ and reducing domains, then, we always have that there is an orbifold chart $\{V'', G'', \pi''\}$ for an open neighborhood of U'' of p such that G'' acts linearly on V'' and $p = \pi''(0)$. In particular, G'' is the isotropy subgroup of p. Throughout, we will use the convention that, for a given point $p \in Q$, a chart labeled $\{V_p, G_p, \pi_p\}$ has these properties with respect to p. We refer to such a chart as an **orbifold chart at** p (see [17] for details).

In the same manner, we can define the cotangent bundle, its exterior powers, etc. of an orbifold Q. Differential forms are integrated over orbifolds in the following sense: If ω is a differential form on Q whose support is contained in a uniformized set U, then the integral $\int_Q \omega$ is defined to be

$$\frac{1}{|G|} \int_V \tilde{\omega},$$

where $\tilde{\omega}$ is the pullback of ω via the projection π. Note that this integral does not depend on the choice of orbifold chart. The integral of a globally-defined differential form is then defined in the same way, using a partition of unity subordinate to a cover consisting of uniformized sets (see [20]).

2.2 The Dimension of a Singularity

In this section, we are interested in understanding the behavior of vector fields on orbifolds and orbifolds with boundary. The primary restriction will be the dimension changes in the space of tangent vectors, the maximal vector space contained in a fiber of the orbifold tangent bundle.

2.2.1 Motivation and Definition

Throughout this section, let Q be an n-dimensional reduced orbifold, $p \in Q$, and let $\{V_p, G_p, \pi_p\}$ be an orbifold chart at p. As was noted above, the tangent bundle TQ of Q is not generally a vector bundle. Indeed, if p is a singular point of Q (i.e. $G_p \neq 1$), then letting $\rho : TQ \to Q$ again denote the projection, $\rho^{-1}(p) \cong \mathbb{R}^n / \{g_\gamma(\tilde{p}) : \gamma \in G\}$ for any lift $\tilde{p}$ of p in V_p; the fiber is not a vector space (recall that $g_\gamma(\tilde{p})$ denotes the infinitesimal action of $\gamma \in G_p$ on $T_{\tilde{p}} V_p$; see Subsection 2.1.3). This fiber, however, is larger than the space of tangent vectors at p (the vectors in $T_{\tilde{p}} V_p$ which are fixed by g_γ for each $\gamma \in G_p$), i.e. the largest vector space contained in $\rho^{-1}(p)$. For a vector field X on Q, $X(p)$ is always an element of the maximal vector space contained in $T_p Q$.

Therefore, the space of tangent vectors of Q at p may, as a vector space, have a smaller dimension than the dimension n of the orbifold. This will play an important role in understanding the zeros of vector fields on orbifolds. In particular, if the space of tangent vectors has dimension 0 at any point, then any vector field must clearly vanish at that point; this is much different than the case of a manifold, and motivates the following definition.

Definition 2.2.1 (Dimension of a Singularity) With the setup as above, we say that $p \in Q$ has **singular dimension** k (or that p is a **singularity of dimension** k) if the space of tangent vectors at p has dimension k.

First, we verify that this definition is well-defined.

Proposition 2.2.2 The singular dimension of a point p does not depend on the choice of the orbifold chart, nor on the choice of the lift $\tilde{p}$ of p in the chart.

Proof:

Fix $p \in Q$, and let $\{V_i, \pi_i, G_i\}$ and $\{V_j, \pi_j, G_j\}$ be orbifold charts (with $U_i := \pi_i(V_i)$ and $U_j := \pi_j(V_j)$ as usual) such that $p \in U_i \cap U_j$. Suppose first that $U_i \subseteq U_j$, and then by the definition of an orbifold, there is an injection λ_{ij} with embedding $\phi_{ij} : V_i \to V_j$.

Now, let $\tilde{p}_i$ be a point in V_i such that $\pi_i(\tilde{p}_i) = p$, and then $\tilde{p}_j := \phi_{ij}(\tilde{p}_i)$ has the property that $\pi_j(\tilde{p}_j) = p$ (by the definition of ϕ_{ij}). With respect to the chart V_i, the singular dimension of p is the dimension of the space of tangent vectors at p; i.e. the dimension of the vector subspace of $T_{\tilde{p}_i} V_i$ which is fixed by the infinitessimal $I_{\tilde{p}_i}$-action. We have that ϕ_{ij} is a diffeomorphism of V_i onto an open subset of V_j. Moreover, as the groups $I_{\tilde{p}_i}$ and $I_{\tilde{p}_j}$ are isomorphic, we have that the associated injective homomorphism

$$f_{ij} : G_i \to G_j$$

maps $I_{\tilde{p}_i}$ onto $I_{\tilde{p}_j}$. Hence, for any $v \in T_{\tilde{p}_i} V_i$ such that $g_\gamma(\tilde{p}_i)v = v$ for every $\gamma \in I_{\tilde{p}_i}$, it is clear that $g_\gamma(\tilde{p}_i) d(\phi_{ij})_{\tilde{p}_i}(v) = d(\phi_{ij})_{\tilde{p}_i}(v)$ for every $\gamma \in I_{\tilde{p}_j}$, and conversely. Therefore, the map $d(\phi_{ij})_{\tilde{p}_i}$ restricts to a vector space isomorphism of the fixed-point set of the $I_{\tilde{p}_i}$-action on $T_{\tilde{p}_i} V_i$ onto the fixed-point set of the $I_{\tilde{p}_j}$-action on $T_{\tilde{p}_j} V_j$, ensuring that their dimensions are equal.

That the singular dimension at p does not depend on our choice of the lift $\tilde{p}$ of p is now clear: if $\tilde{p}_i'$ is another choice of a lift in V_i, then there is an injection $\lambda_{ii} = (\phi_{ii}, f_{ii})$ of (V_i, G_i, π_i) into itself such that $\phi_{ii}(\tilde{p}_i) = \tilde{p}_i'$. Hence we may apply the above argument.

In the case where U_i is not a subset of U_j, there is a chart over some set U_k with $p \in U_k \subseteq U_i \cap U_j$. The above argument gives us that the dimension is the same with respect to U_k as with respect to U_i, and the same with respect to U_k as with respect to U_j.

Q.E.D.

2.2.2 Properties

Note that every point in Q has a singular dimension, not simply the singular points. In particular, we have

Proposition 2.2.3 The non-singular points in Q are precisely the points with singular dimension n, where n is the dimension of Q.

Proof:

Let p be a point in Q, and suppose that p has singular dimension n. Let $\{V_p, G_p, \pi_p\}$ be an orbifold chart at p. Taking a lift $\tilde{p} \in V_p$ such that $\pi(\tilde{p}) = p$, we then have by hypothesis that the space of $I_{\tilde{p}}$-invariant vectors in $T_{\tilde{p}} V_p$, has dimension n, and hence that $I_{\tilde{p}}$ acts trivially on $T_{\tilde{p}} V_p$. Of course, as Q is reduced, and as $G_p = I_{\tilde{p}}$ by our choice of charts, this implies that $G_p = 1$, and that p is not a singular point.

Conversely, for p a non-singular point in Q, $I_{\tilde{p}} = 1$ for any lifting $\tilde{p}$ of p into any chart, and hence fixes the entire n-dimensional vector space $T_{\tilde{p}}V$ in the corresponding chart.

Q.E.D.

Recall that Σ_Q denotes the set of all singular points in Q (i.e. points $p \in Q$ such that $G_p \neq 1$ for any chart $\{V_p, G_p, \pi_p\}$ at p). For each k with $k = 0, 1, \ldots, n$, let Σ_k denote the set of all singular points with singular dimension k. Clearly $\Sigma_Q = \bigcup_{k=0}^{n-1} \Sigma_k$. As was pointed out in Satake [20], $Q \backslash \Sigma_Q$ (which, in our notation is Σ_n) is an n-dimensional manifold. However, we also have following proposition (see also [9]):

Proposition 2.2.4 Let Q be an n-dimensional orbifold (with boundary). Then for $k = 0, 1, \ldots, n$, Σ_k naturally has the structure of a k-dimensional manifold (with boundary, and $\partial\Sigma_k = \Sigma_k \cap \partial Q$).

Proof:

Fix k with $0 \leq k \leq n$, let p be a point in Σ_k, and fix an orbifold chart $\{V_p, G_p, \pi_p\}$ at p. Let $\tilde{p} \in V_p$ be a lift of p in the chart. By the definition of Σ_k, we have that the $I_{\tilde{p}}$-invariant subspace of $T_{\tilde{p}}V_p$ is of dimension k. Hence, as G_p acts linearly on V, there is a k-dimensional subspace Y of $\mathbb{R}^n \supset V_p$ on which $I_{\tilde{p}}$ acts trivially.

If $k = 0$, then $Y = \{\tilde{p}\}$ is clearly the only invariant point of V_p, and hence U_p is an open set containing p in which p is the only 0-dimensional singularity. Therefore, the 0-dimensional singularities are isolated and form a 0-manifold. The rest of the proof will deal with the case $k > 0$.

First suppose that p is not a boundary point of Q (so, in particular, we may assume that V_p does not have boundary). Note that by the definition of Y, each point $\tilde{x} \in Y$ is fixed by $I_{\tilde{p}}$. Hence, the isotropy subgroup $I_{\tilde{x}}$ of each $\tilde{x} \in Y$ in G_p clearly

contains $I_{\tilde{p}}$. However, by our choice of charts, $I_{\tilde{p}} = G_p$, so that each $I_{\tilde{x}} = G_p$. Hence, we may take $W = Y \cap V_p$ to be an open neighborhood of $\tilde{p}$ in Y with constant isotropy.

Note that as $W \subset Y$, the restriction $(\pi_p)_{|W}$ of π_p to W is an injective map of W onto $\pi_p(W)$. Hence, as $\pi_p(W) \cap \Sigma_k$ is clearly open in Σ_k, $(\pi_p)_{|W}$ is a $\mathcal{C}^\infty$ bijection of W onto a neighborhood of p in Σ_k. In particular, as W is open in $Y = \mathbb{R}^k$, π_p restricts to a manifold chart for Σ_k near p. That such charts cover Σ_k is clear, as p was arbitrary. Moreover, that such charts intersect with the appropriate transition follows directly from the existence of the injections λ_{ij} and associated ϕ_{ij}.

Similarly, if Q has boundary and $p \in \partial Q$, then Y is a subspace of $\mathbb{R}^n \supset \mathbb{R}^n_+ \supset V_p$, and $W = Y \cap V_p$ is an open subset of $Y \cap \mathbb{R}^n_+$. Hence, we see that π_p restricts to a chart of a manifold with boundary on Σ_k with $p \in \partial\Sigma_k$.

Q.E.D.

Example 2.2.5 We note here that the connected components of the set Σ_Q need not have the structure of a manifold. For example, within a 3-dimensional orbifold, we may have a singular set in the shape of an 8 that fails to be a manifold at one point p. This can happen, for instance, when a chart $\{V_1, G_1, \pi_1\}$ at p has group

$$G_1 = \langle R^x_\pi, R^z_{\frac{2\pi}{3}} \rangle$$

where

$$R^x_\pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

denotes a rotation of π radians about the x-axis, and

$$R^z_{\frac{2\pi}{3}} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) & 0 \\ \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) & 0 \\ \\ 0 & 0 & 1 \end{bmatrix}$$

denotes a rotation of $\frac{2\pi}{3}$ radians about the z-axis. By direct computation, $R^x_\pi R^z_{\frac{2\pi}{3}} (R^x_\pi)^{-1} = (R^z_{\frac{2\pi}{3}})^{-1}$ so that G_1 is isomorphic to the dihedral group D_6. Define further the charts $\{V_2, G_2, \pi_2\}$ and $\{V_3, G_3, \pi_3\}$ covering the rest of Σ_Q with groups

$$G_2 = \langle \mathbb{R}^x_\pi \rangle \cong \mathbb{Z}_2$$

and

$$G_3 = \langle \mathbb{R}^z_{\frac{2\pi}{3}} \rangle \cong \mathbb{Z}_3$$

as pictured (see Figure 2.3).

We note, however, that the set $\Sigma_1 = \Sigma_Q \backslash \{p\}$ is an open 1-manifold diffeomorphic to $(0,1) \sqcup (2,3)$, and the set $\Sigma_0 = \{p\}$ is a 0-manifold.

This example serves to illutrate the structure of the set Σ_Q almost in general. For, intuitively, in any chart $\{V, G, \pi\}$, vectors tangent to V which are invariant under the $\{g_\gamma : \gamma \in G\}$-action must clearly be tangent to the singular set Σ_Q. Hence, at points where two singular strata 'run into' one another, the dimension of the space of tangent vectors decreases. The intersection of the closures of two connected components of Σ_k, for some k, belong to Σ_j for some $j < k$.

We state the following obvious corollaries to Proposition 2.2.4 and its proof.

Corollary 2.2.6 With the notation as above, Σ_0 is a set of isolated points. In particular, if Q is compact, Σ_0 is finite.

Corollary 2.2.7 Let $p \in \Sigma_k \subset Q$. Then the set of tangent vectors of Q at p is canonically identified with $T_p\Sigma_k$.

2.2.3 Examples of 0-Dimensional Singularities

Our primary interest here is the case of singularities with dimension zero, over which the largest vector space contained in T_pQ is $\{\mathbf{0}\}$. For over such points, any vector field must vanish.

Example 2.2.8 Consider, for example, the cone C, which is the quotient of $\mathbb{R}^2$ by the group of rotations

$$G = \langle \gamma \rangle = \mathbb{Z}_3$$

where

$$\gamma := \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix}$$

(see Figure 2.4).

This is clearly an orbifold with one chart where $V = \mathbb{R}^2$, $G \cong \mathbb{Z}_3$, and π is the quotient map induced by the group action; and one singular point $p = \pi(0,0)$. A vector field on C is precisely a G-invariant vector field on $V = \mathbb{R}^2$. However, as γ has no nontrivial eigenvectors, any vector on the tangent space at $(0,0)$ is not fixed by G, so that any vector field on C must vanish at the singular point.

It may be helpful if we develop the tangent space of this previous example explicitly. Any point $q \in C$, $q \neq p$, with lift $\tilde{q} \in V$ has trivial isotropy group $I_{\tilde{q}} = 1$. Hence, the action of $I_{\tilde{q}}$ on the tangent space $T_{\tilde{q}}V \cong \{\tilde{q}\} \times \mathbb{R}^2$ is trivial, and the space of tangent

vectors is

$$\begin{aligned} T_qQ &= \pi^*[(T_{\tilde{q}}V)^{I_{\tilde{q}}}] \\ &= \pi^*(T_{\tilde{q}}V) \\ &\cong \mathbb{R}^2. \end{aligned}$$

The set $\Sigma_2 = C \backslash \{p\}$ is clearly a smooth 2-dimensional manifold. Moreover, taking a point $q \in \Sigma_2$ and an open ball V' about q which does not intersect $\tilde{p} = (0,0)$ (small enough so that $\forall \gamma \in G, (\gamma V') \cap V' = \emptyset$), the restriction of π to V' gives a manifold chart of Σ_2 near q, and the manifold tangent space of Σ_2 at q is exacly the orbifold tangent space of C at q.

Now, the point $\tilde{p} = (0,0)$ has isotropy group $I_{\tilde{p}} = G$. Identifying $T_{\tilde{p}}V$ with $\mathbb{R}^2$ in the usual way, we have that as G is linear, and $g_\gamma(\tilde{p}) = \tilde{p}$ (identifying coordinates on V with coordinates on $T_{\tilde{p}}V$ via the exponential map; this simply states that the Jacobian of a linear operator is itself). Then $(T_{\tilde{p}}V)^{I_{\tilde{p}}}$ is defined to be the set of vectors in $T_{\tilde{p}}V \cong \mathbb{R}^2$ which are invariant under the action of the group generated by

$g_\gamma(\tilde{p}) = g = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix}$, which is clearly only the zero vector. Hence, $T_pQ = \pi^*(T_{\tilde{p}}V^{I_{\tilde{p}}}) = \pi^*(\{\mathbf{0}\}) = \{\mathbf{0}\}$, and the space of tangent vectors to C at p contains only the zero vector.

The following example is meant to illustrate the limitations of Satake's definition of an orbifold with boundary (see [20] for the definition). In particular, it is an example in which there is no non-vanishing vector field on the boundary.

Example 2.2.9 Now consider the orbifold C_0 with boundary (using Satake's definition

of an orbifold with boundary) defined to be the union of the following subsets of C:

$$V_1 := \left\{(\rho,\theta) : \rho \geq 0, \tfrac{-\pi}{6} \leq \theta \leq \tfrac{\pi}{6}\right\},$$

$$V_2 := \left\{(\rho,\theta) : \rho \geq 0, \tfrac{3\pi}{6} \leq \theta \leq \tfrac{5\pi}{6}\right\}, \text{ and}$$

$$V_3 := \left\{(\rho,\theta) : \rho \geq 0, \tfrac{7\pi}{6} \leq \theta \leq \tfrac{9\pi}{6}\right\}$$

(we use standard polar coordinates (ρ, θ) on $\mathbb{R}^2$; see Figure 2.5).

Then

$$C_0 := \pi(V_1) \cup \pi(V_2) \cup \pi(V_3)$$

is clearly an orbifold with boundary containing one singular point $p = \pi(0, 0)$ of dimension zero. Hence, as $p \in \partial C_0$, there is no vector field X on Q_0 which is non-vanishing on the boundary. Indeed, for any such vector field, $X(p) = \mathbf{0}$. Note further that C_0 does not admit a product structure near the boundary; i.e. there is no open neighborhood of the boundary in Q diffeomorphic to $[0, \epsilon) \times \partial Q$.

We should note that these two previous examples are non-compact for simplicity of exposition, but that, by reducing the domain of the charts and patching them together with charts where G is the trivial group, the same type of singularity can clearly occur in the case of a compact orbifold: the first as the sole singularity in the $\mathbb{Z}_3$-teardrop, and the second in a teardrop which is missing a piece homeomorphic to a disc, where the singularity occurs on the boundary.

For our definition of orbifolds, however, the above cannot occur. Indeed, it is trivial to show (using a chart with linear group action) that if Q is an orbifold with boundary M, then a neighborhood of M in Q is diffeomorphic to $[0, \epsilon) \times M$). Therefore, for every point p on the boundary of Q, the tangent space T_pQ contains a one-dimensional subspace independent from T_pM. In particular, although M may, as an orbifold, contain 0-dimensional singularities, as a subset of Q, it does not. As well, by Corollary 2.2.6,

on a compact orbifold, we need not worry about the existence of a vector field with a finite number of zeros.

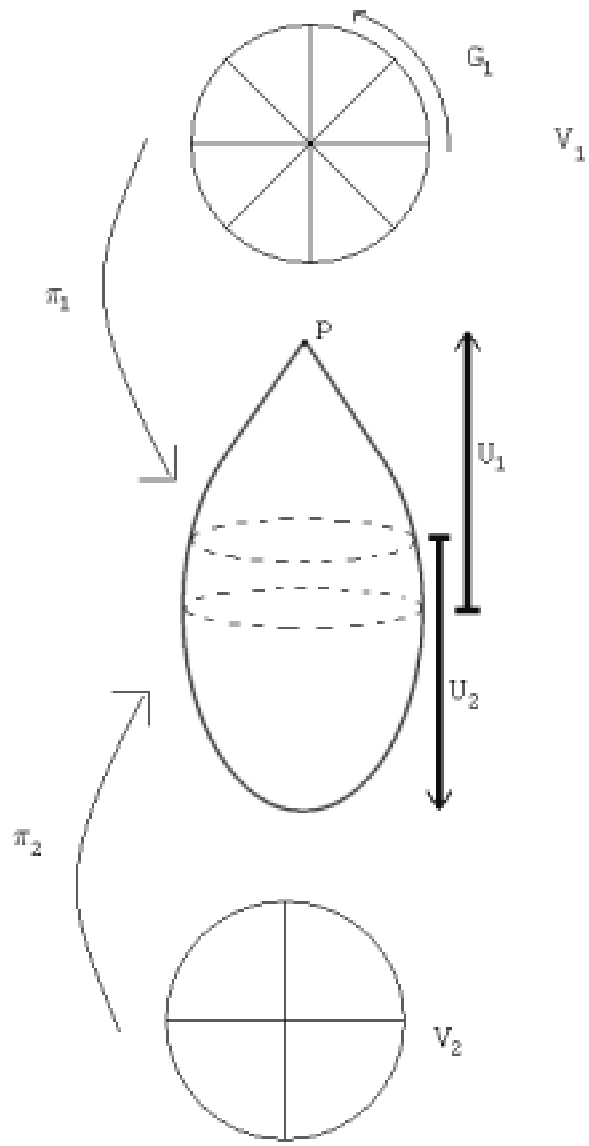

Figure 2.1: The $\mathbb{Z}_k$-teardrop.

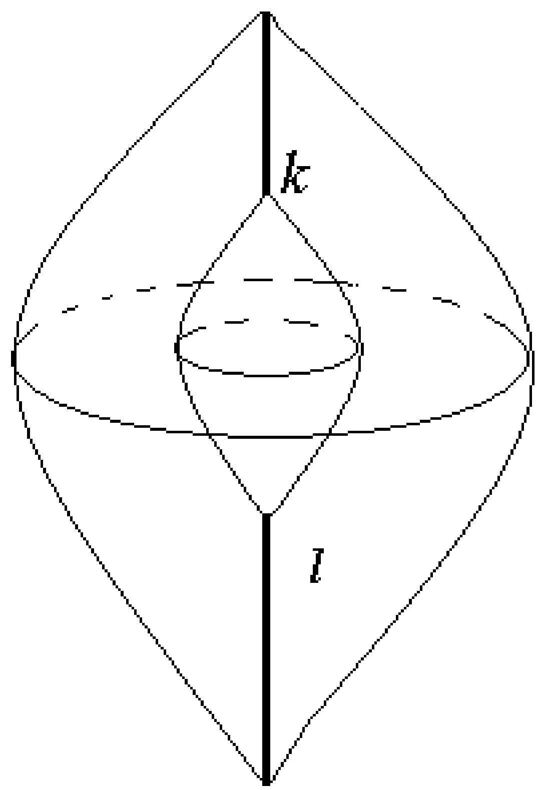

Figure 2.2: The $\mathbb{Z}_k$-$\mathbb{Z}_l$-solid hollow football is the region between the two closed 2-dimensional orbifolds.

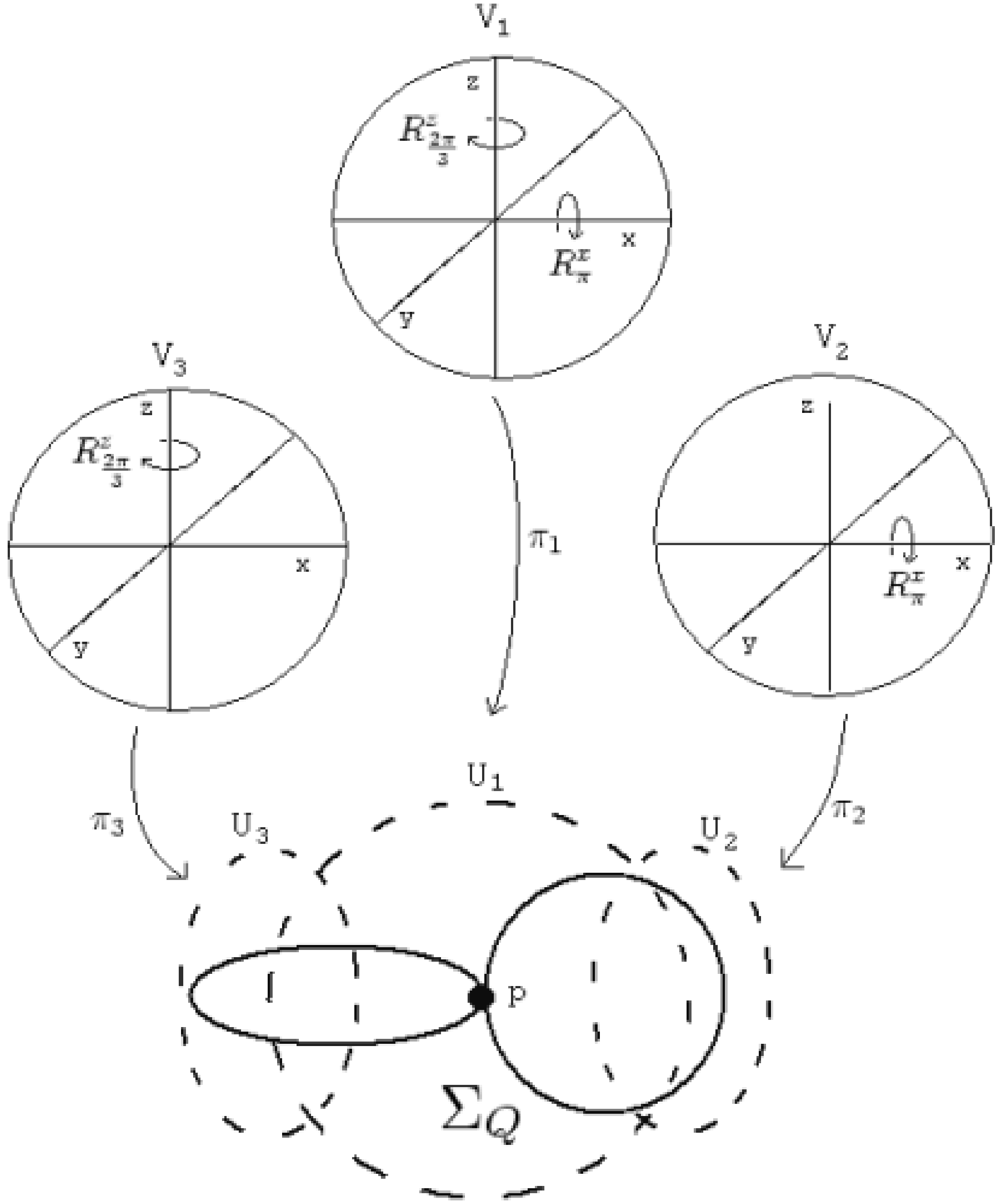

Figure 2.3: A singular set Σ_Q in a 3-dimensional orbifold which is not a manifold.

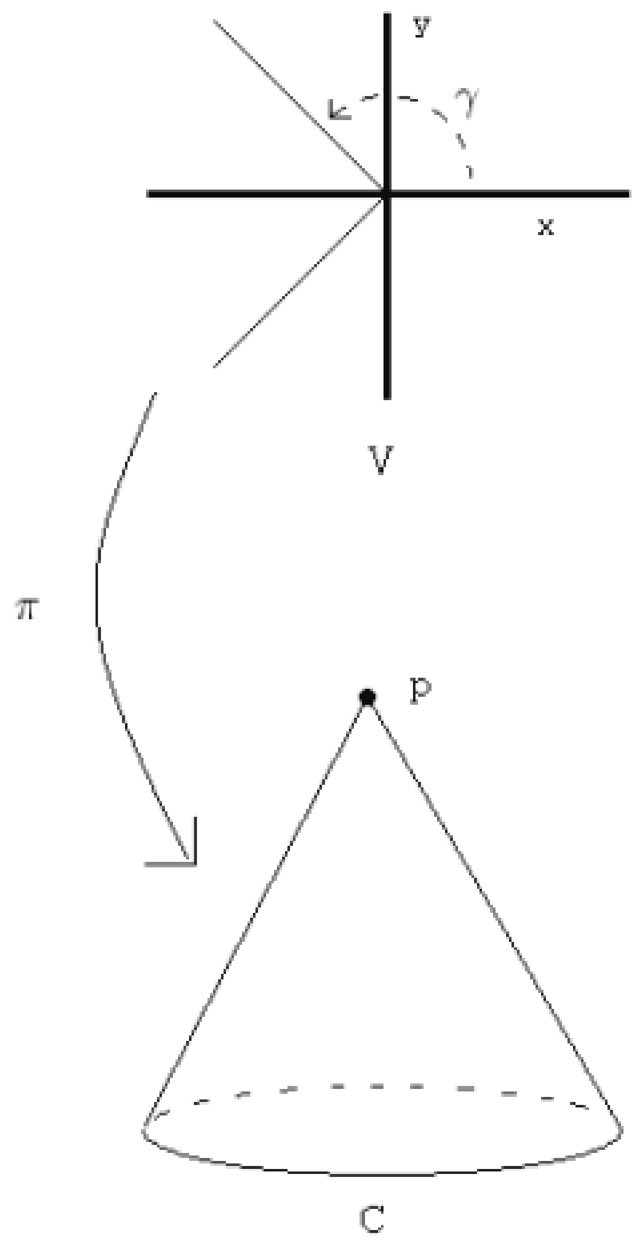

Figure 2.4: The cone $C = \mathbb{R}^2/\mathbb{Z}_3$.

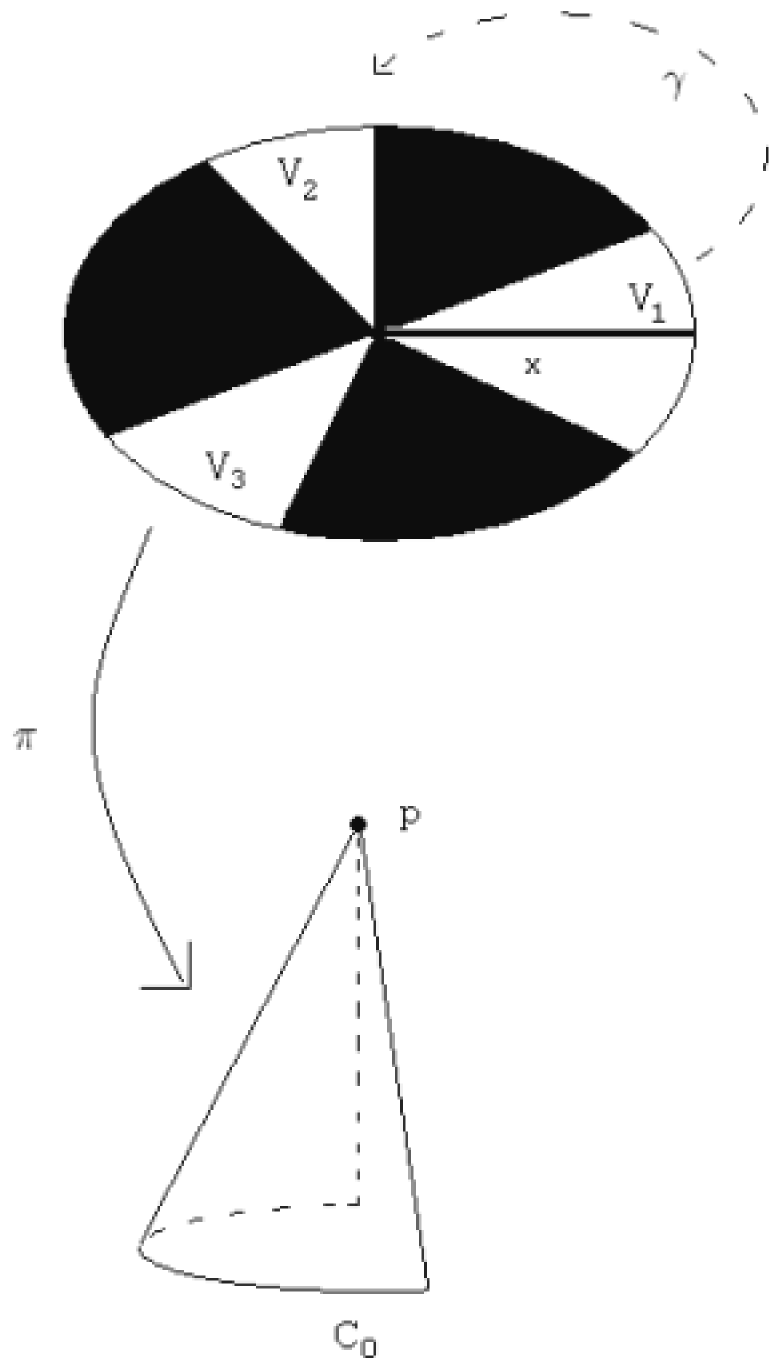

Figure 2.5: A dimension-0 singularity on the boundary of an orbifold (using Satake's original definition of an orbifold with boundary).

Chapter 3

The First Gauss-Bonnet and Poincaré-Hopf Theorems for Orbifolds With Boundary

3.1 Introduction

Our goal in this section is to generalize Satake's Gauss-Bonnet Theorem for orbifolds with boundary [20] using the more modern definition of an orbifold with boundary. We will see that, with this definition, Satake's boundary term can be simplified considerably (indeed, in some cases it will vanish).

To this end, a note is necessary. The statemenet of Satake's Gauss-Bonnet Theorem for orbifolds with boundary refers to an outward-pointing unit normal vector field on the boundary M of Q. In the case that Q is taken to be an orbifold with boundary as defined by Satake, the boundary may very well contain 0-dimensional singular points (as in the case of the sliced cone in Example 2.2.9), in which case this vector field does not exist—indeed, in such a case there will be no non-vanishing vector field on the boundary. However, our definition does not allow such behavior.

This section follows constructions in [4], [20], and [22]. However, our notation primarily follows that of [22]. Where possible, we state our results and constructions for general orbifold vector bundles, though our primary application will be to the tangent bundle. In particular, many of the definitions will be made in as a general a context as possible to facilitate the proof of the Thom Isomorphism Theorem for Orbifolds (Theorem 3.3.8) in the general case. All orbifolds under consideration are assumed to

be reduced.

3.2 The Results on the Level of Differential Forms

3.2.1 The Setup

In this subsection, we fix the context and notation for the rest of the section. Let Q be a compact, connected, orientable orbifold of dimension n with boundary $\partial Q =: M$, taking the orientation of M to be the orientation inherited from Q with respect to the outer unit normal vector field. We will use with liberty the fact that that there is a neighborhood of M in Q diffeomorphic to $[0, \epsilon) \times M$. Let $\rho : E \to Q$ be an orbifold vector bundle over Q of rank $l = 2m$ or $2m+1$ equipped with a Euclidean metric. Let SE denote the unit sphere bundle of E with respect to the metric, with projection still denoted ρ. Fix a compatible $SO(n)$-connection ω with curvature Ω on E. With respect to a local orthonormal frame $(e_1, \ldots, e_l)$ for E, we let $(u_1, \ldots, u_l)$ denote the component functions on E and $(\theta_1, \ldots, \theta_l)$ the basic forms: $\theta := du + \omega u$.

For the specific case where $E = TQ$ is the tangent bundle of Q, we require that the metric respect the product structure of Q near the boundary. In particular, at any point $p \in M$, with respect to any chart $\{V, G, \pi\}$ and any oriented orthonormal frame field $(e_1, e_2, \ldots, e_n)$ for the fiber $(\pi^* TQ)_{\tilde{p}}$ at $\tilde{p}$ ($\pi(\tilde{p}) = p$ as usual), we have that $\omega_{i,j} = \Omega_{i,j} = 0$ whenever $i = 1$ or $j = 1$. Note that in the case of the tangent bundle, $l = n$.

Let X be a vector field on Q which is non-zero on M and has a finite number of fixed points $p_1, \ldots, p_s$ on the interior of Q. Denote by $B_r(p_i)$ the geodesic ball about p_i of radius r, and for ease of notation, let $B_r(P) := \bigcup_{i=1}^{s} B_r(p_i)$. Let $\alpha : Q \backslash \{p_1, \ldots, p_s\} \to ST$ denote the section of the sphere bundle STQ induced by X. We will sometimes require that X extend the outer unit normal vector field on M, in which case we will use the notation X_0 for the vector field and α_0 for the induced section of the sphere bundle.

3.2.2 Notes on the Definitions of the Integrands

A note is in order on our definition of the Euler curvature form $E(\Omega)$ and its secondary form Ψ. These forms, which were originally defined by Chern [4] in the case of the tangent bundle of a closed Riemannian manifold of even dimension, were used by Satake [20] and Sha [22] in generalizations of the Gauss-Bonnet Theorem to the case of closed orbifolds and of the Poincaré-Hopf Theorem to the case of manifolds with boundary, respectively. However, the definitions of the two forms differ slightly, primarily due to Chern's further developments in [5]. In particular, in [5], Chern extended the definition of these forms to odd-dimensional manifolds and reversed the sign of the Euler curvature form. Hence, some care must be taken with respect to how the various definitions of the forms fit together.

In [4], the forms first appeared on the tangent bundle of a closed Riemannian manifold of dimension $n = 2m$. They were called Ω and Π, respectively, and defined as follows with respect to a frame $(e_1, \ldots, e_n)$ (again with component functions $(u_1, \ldots, u_n)$ and basic forms $(\theta_1, \ldots, \theta_n)$):

$$\Omega := (-1)^{m-1} \frac{1}{2^{2m}\pi^m m!} \sum_{\tau \in S(n)} (-1)^\tau \Omega_{\tau(1)\tau(2)} \wedge \cdots \wedge \Omega_{\tau(n-1)\tau(n)}$$

is the Euler curvature form, and

$$\Pi := \frac{1}{\pi^m} \sum_{k=0}^{m-1} \frac{1}{1 \cdot 3 \cdots (n-2k-1)} \Phi_k$$

its secondary form on the unit sphere bundle, where

$$\Phi_k = \sum_{\tau \in S(n)} (-1)^\tau u_{\tau(1)} \theta_{\tau(2)} \wedge \cdots \wedge \theta_{\tau(n-2k)} \wedge \Omega_{\tau(n-2k+1)\tau(n-2k+2)} \wedge \cdots \wedge \Omega_{\tau(n-1)\tau(n)}$$

for $k = 0, 1, \ldots, m-1$ (as usual, $S(n)$ denotes the group of permutations of n letters).

In [5], Chern introduced a new definition of the form Π on both even and odd-

dimensional manifolds:

$$\Pi := \begin{cases} \frac{1}{\pi^m} \sum_{k=0}^{m-1} (-1)^k \frac{1}{1\cdot 3 \cdots (n-2k-1) 2^{m+k} \cdot k!} \Phi_k, & \text{if } n = 2m \text{ is even,} \\ \\ \frac{1}{2^n \pi^m m!} \sum_{k=0}^{m} (-1)^{k+1} \binom{m}{k} \Phi_k & \text{if } n = 2m+1 \text{ is odd,} \end{cases}$$

where

$$\Phi_k = \sum_{\tau \in S(n)} \Omega_{\tau(1)\tau(2)} \wedge \cdots \wedge \Omega_{\tau(2k-1)\tau(2k)} \wedge \omega_{\tau(2k+1)n} \wedge \cdots \wedge \omega_{\tau(n-1)n}$$

for $k = 0, 1, \ldots, \left[\frac{n}{2}\right] - 1$. It is pointed out ([5], page 675) that when n is even, the Φ_k, and hence Π, reduce to their definitions in [4]. As was mentioned above, the definition of Ω differs from the previous only by a minus sign (so that a sign is introduced in the relationship between the two forms).

The reason for our difficulty is that various authors have since used different conventions regarding the signs of these forms, so that the definitions of the forms are by no means standard. The definitions of Chern in [5] are such that, in the odd dimensional case, the integral of the secondary form on a fiber of the sphere bundle is -1 in the odd-dimensional case, and hence relates to the negative index of the vector field. This, of course, offers no difficulty for closed manifolds, as the index of the vector field is in this case always zero. However, in the case of manifolds with boundary, this leads to a negative sign in the formula, which is undesirable. The reader who may compare our computations to those in [4], [5], [20], [21], or [22] is **warned** to compare the definitions carefully and take into account the appropriate sign conventions.

We will follow Sha's notation for the most part, letting $E(\Omega)$ denote the Euler curvature form (with respect to the connection Ω) and Ψ its secondary form. Our sign conventions are chosen so that in the resulting formula, the index of the vector field has the same sign in both the even and odd cases. Hence, we will use the following definitions, for an arbitrary vector bundle of rank l (again, $l = n$ in the case of the

tangent bundle):

$$E(\Omega) := \begin{cases} \frac{1}{2^{2m}\pi^m m!} \sum\limits_{\tau\in S(l)} (-1)^\tau \Omega_{\tau(1)\tau(2)} \wedge \cdots \wedge \Omega_{\tau(l-1)\tau(l)}, & \text{if } l = 2m \text{ is even,} \\ \\ 0, & \text{if } l = 2m+1 \text{ is odd} \end{cases}$$

is the Euler curvature form, which agrees with the definition of Sha [22]. The secondary form on the unit sphere bundle is

$$\Psi = \begin{cases} \frac{(-1)^m}{\pi^m} \sum\limits_{k=0}^{m-1} (-1)^k \frac{1}{1\cdot 3\cdots(l-2k-1)2^{m+k}\cdot k!} \Phi_k, & \text{if } l = 2m \text{ is even,} \\ \\ \frac{-1}{2^l \pi^m m!} \sum\limits_{k=0}^{m} \binom{m}{k} \Phi_k, & \text{if } l = 2m+1 \text{ is odd,} \end{cases}$$

where

$$\Phi_k = \sum_{\tau\in S(l)} (-1)^\tau u_{\tau(1)} \theta_{\tau(2)} \wedge \cdots \wedge \theta_{\tau(l-2k)} \wedge \Omega_{\tau(l-2k+1)\tau(l-2k+2)} \wedge \cdots \wedge \Omega_{\tau(l-1)\tau(l)}.$$

We have that on the unit sphere bundle, $d\Psi = -\rho^* E(\Omega)$, where $\rho : Q \to SE$ again denotes the bundle projection. Moreover, we have that $\int\limits_{SE_p} \Psi = \frac{1}{|I_p|}$ where p is any point in Q with isotropy I_p. Of course, these relations are preserved up to the sign using the definitions in any of our references (in the case of a manifold, $\frac{1}{|I_p|}$ is always equal to 1).

3.2.3 The Result of Satake for Orbifolds With Boundary

In [20], we have the following Gauss-Bonnet Theorem.

Theorem 3.2.1 (Satake) Let Q be an oriented compact Riemannian orbifold with boundary M, let $\mathcal{N}$ be the outward-pointing unit normal vector field on M, and let α_0 be the induced section of $STQ_{|M}$. Then we have

$$\int\limits_Q E(\Omega) = \chi'_{orb}(Q) - \int\limits_M \alpha_0^* \Psi,$$

where M is oriented by the induced orientation with respect to the outer normal vector field on M.

It is noted that the requirement of orientability can be lifted by using the standard reduction to the orientable double-cover.

Here, $\chi_{orb}(M)$ and $\chi'_{orb}(Q)$ are the **orbifold Euler characteristic** and **inner orbifold Euler characteristic** of M and Q, respectively, whose definitions we recall (see Appendix A for a discussion of alternate Euler characteristics for orbifolds).

Satake's definition of the Euler characteristic for a closed orbifold without boundary came from his proof of the Gauss-Bonnet Theorem in this case, with the equation

$$\int_Q E(\Omega) = \sum_{i=1}^{s} \mathrm{ind}_X(p_i).$$

Since the left-hand side does not depend on the vector field X, and the right-hand side does not depend on the metric, this number is an invariant of the orbifold itself. We denote this invariant $\chi_{orb}(Q)$ and refer to it as the **orbifold Euler Characteristic**. Satake goes on to show how this number can be computed in terms of a suitable triangulation, the existence of such a triangulation since having been demonstrated in [15]. Specifically, if $\mathcal{T}$ is a triangulation of Q such that the order of the isotropy group is a constant function on the interior of any simplex $\sigma \in \mathcal{T}$, then letting N_σ denote this order, we have

$$\chi_{orb}(Q) = \sum_{\sigma \in \mathcal{T}} \frac{(-1)^{\dim \sigma}}{N_\sigma}.$$

Now, in the case that Q has boundary, the **inner orbifold Euler Characteristic** $\chi'_{orb}(Q)$ is defined similarly, but with respect to a particularly chosen vector field: one which extends the outward unit normal $\mathcal{N}$ on the boundary M of Q. In this case, given a simplicial decomposition as above, we have

$$\chi'_{orb}(Q) = \sum_{\sigma \in \mathcal{T}_0} \frac{(-1)^{\dim \sigma}}{N_\sigma},$$

where $\mathcal{T}_0$ denotes the collection of simplices which are not completely contained in the boundary.

3.2.4 Computations of $\alpha^*(\Psi)$ in the Case That X Extends the Outer Unit Normal Vector Field on M

As was pointed out in Sha [22] for the case of manifolds, in the specific case that the vector field X_0 extends the outward-pointing normal vector field $\mathcal{N}$ and n is even, if α_0 denotes the section of STQ induced by X_0, we have that

$$\alpha_0^*(\Psi) = 0$$

on M. This is proven as follows:

With a chart $\{V, G, \pi\}$ and an orthonormal oriented frame field $(e_1, e_2, \ldots, e_n)$ chosen for a point $\tilde{p}$ on the boundary $\pi^{-1}(M) = \partial V$ such that the $(e_2, \ldots, e_n)$ are tangent to the boundary of V, the form Ψ is a sum of terms of the form

$$\Phi_k = \sum_{\tau \in S(n)} (-1)^\tau u_{\tau(1)} \theta_{\tau(2)} \wedge \cdots \wedge \theta_{\tau(n-2k)} \wedge \Omega_{\tau(n-2k+1)\tau(n-2k+2)} \wedge \cdots \wedge \Omega_{\tau(n-1)\tau(n)}$$

for $k = 0, 1, \ldots m-1$. We have that the $\mathcal{N}$ is locally equal to $-e_1$ in the lift to V; it has coordinates $(u_1, u_2, \ldots, u_n) = (-1, 0, \ldots, 0)$ with respect to this frame. Hence, as $\theta = du + \omega u$, we have that $\alpha_0^*(\theta_i)$ vanishes for each $i > 1$. Note that each term either has a factor of θ_i for $i > 1$, or has a factor of θ_1 and a factor of u_i for $i > 1$, in which case $u_i = 0$. Therefore, each of these terms vanish when composed with α_0, and hence

$$\alpha_0^*(\Psi) = 0.$$

It will be worthwhile to see how these computations simplify in the case of $n = 2m+1$ odd (this is also noted by Sha in [22]). As above, all of the θ_i factors vanish

except for θ_1, but there does exist one Ψ_k which does not contain any such factors:

$$\begin{aligned}\Phi_m &= \sum_{\tau \in S(n)} (-1)^\tau u_{\tau(1)} \Omega_{\tau(n-2m+1)\tau(n-2m+2)} \wedge \cdots \wedge \Omega_{\tau(n-1)\tau(n)} \\ &= \sum_{\tau \in S(n)} (-1)^\tau u_{\tau(1)} \Omega_{\tau(2)\tau(3)} \wedge \cdots \wedge \Omega_{\tau(n-1)\tau(n)}.\end{aligned}$$

Moreover, in $\alpha_0^*(\Phi_m)$, we have that the coefficient $u_{\tau(1)} = 0$ in every term except those such that $\tau(1) = 1$ (recall that $u_1 = -1$ and $u_i = 0$ for $i > 1$). Hence,

$$\begin{aligned}\alpha_0^*(\Phi_m) &= u_1 \sum_{\tau \in S(n-1)} (-1)^\tau \Omega_{\tau(2)\tau(3)} \wedge \cdots \wedge \Omega_{\tau(n-1)\tau(n)} \\ &= -\sum_{\tau \in S(n-1)} (-1)^\tau \Omega_{\tau(2)\tau(3)} \wedge \cdots \wedge \Omega_{\tau(n-1)\tau(n)},\end{aligned}$$

where $S(n-1)$ is understood to be the group of permutations on $\{2, 3, \ldots, n\}$. So in this case,

$$\begin{aligned}\alpha_0^*(\Psi) &= \tfrac{-1}{2^n \pi^m m!} \tbinom{m}{m} \alpha_0^*(\Phi_m) \\ &= \tfrac{-1}{2^n \pi^m m!} \left[-\sum_{\tau \in S(n-1)} (-1)^\tau \Omega_{\tau(2)\tau(3)} \wedge \cdots \wedge \Omega_{\tau(n-1)\tau(n)} \right] \\ &= \tfrac{1}{2} \left[\tfrac{1}{2^{2m} \pi^m m!} \sum_{\tau \in S(n-1)} (-1)^\tau \Omega_{\tau(2)\tau(3)} \wedge \cdots \wedge \Omega_{\tau(n-1)\tau(n)} \right]\end{aligned}$$

Recall that $(e_2, \ldots, e_n)$ is a frame field for TM, and that M is of dimension $n-1 = 2m$. Therefore, the above form is precisely $\frac{1}{2}$ times the Euler curvature form for TM. In summary,

$$\alpha_0^*(\Psi) = \frac{1}{2} E(\Omega_{|M}).$$

Hence, in the odd case,

$$\begin{aligned}\int_M \alpha_0^*(\Psi) &= \int_M \tfrac{1}{2} E(\Omega_M) \\ &= \tfrac{1}{2} \chi_{orb}(M),\end{aligned}$$

the last equality following from Satake's Gauss-Bonnet Theorem for closed orbifolds.

3.2.5 The Theorems for Orbifolds With Boundary

With this, we may restate Satake's result for orbifolds with boundary.

Theorem 3.2.2 (The First Gauss-Bonnet Theorem for Orbifolds with Boundary) Let Q be a compact orbifold of dimension n with boundary M, and let $E(\Omega)$ be defined as above in terms of the curvature Ω of a connection ω. Then

$$\int_Q E(\Omega) = \begin{cases} \chi'_{orb}(Q), & n = 2m, \\ \\ \chi'_{orb}(Q) - \frac{1}{2}\chi_{orb}(M), & n = 2m+1. \end{cases}$$

Note that as $E(\Omega)$ is defined to be zero in the case that n is odd, we have the familiar relation

$$\chi'_{orb}(Q) = \frac{1}{2}\chi_{orb}(M).$$

Example 3.2.3 For example, the solid $\mathbb{Z}_3$-football (i.e. the closed ball in $\mathbb{R}^3$ with the usual action of $\mathbb{Z}_3$ via rotations; see [2]) is an orbifold with boundary whose inner orbifold Euler Characteristic is $\frac{1}{3}$. Its boundary has orbifold Euler Characteristic $\frac{2}{3}$. This can be easily verified with a simplicial decomposition, or using the fact that the the space is diffeomorphic to $\overline{\mathbb{D}_3}/\mathbb{Z}_3$ and the boundary $S^2/\mathbb{Z}_3$.

We are now in the position to extend the Poincaré-Hopf Theorem to the case of a compact oriented orbifold with boundary. Begin with the setup given in Section 3.2.1 with $E = TQ$ and X a vector field with a finite number of zeros $p_1, \ldots, p_s$ on the interior of Q. We require only that X does not vanish on the boundary. The index of X at p_i is defined in a manner analogous to the integral; if $\{V_{p_i}, G_{p_i}, \pi_{p_i}\}$ is a chart at p_i and $\tilde{X}$ denotes the lift of X to V_{p_i}, then the index of X at p is

$$\mathrm{ind}_X(p_i) := \frac{1}{|G_{p_i}|}\mathrm{ind}_{\tilde{X}}(\tilde{p}_i),$$

(see [20]). Of course, $\frac{1}{|G_{p_i}|} = \frac{1}{|I_{p_i}|}$ for this choice of chart; i.e. G_{p_i} is the isotropy group of p_i.

In both the even and odd cases, we have that

$$d\Psi = -\rho^* E(\Omega)$$

on the unit sphere bundle of the tangent bundle. Moreover, at each singular point p_i, we have that

$$\lim_{r \to 0^+} \int\limits_{\partial B_r(p_i)} \alpha^* \Psi = -\mathrm{ind}_X(p_i).$$

This follows from the fact that the integral $\int\limits_{ST_pQ} \Psi$ of Ψ over any fiber ST_pQ of the unit sphere bundle is $\frac{1}{|I_p|}$. The minus sign is due to the fact that the orientation that $\partial B_r(p_i)$ inherits as a component of the boundary of $Q \backslash B_r(P)$ is the opposite orientation of that used in definition of the index (see for example [8] or [13]). Recall that $B_r(P)$ denotes the union $\bigcup\limits_{i=1}^{s} B_r(p_i)$, and set $\mathrm{ind}(X) := \mathrm{ind}_X(P) := \sum\limits_{i=1}^{s} \mathrm{ind}_X(p_i)$.

With this, based on Sha's proof of the Poincaré-Hopf Theorem for manifolds with boundary, we have the following:

If $n = 2m$ is even, then

$$\begin{aligned}
\chi'_{orb}(Q) &= \int_Q E(\Omega) \\
&\quad \text{(by Theorem 3.2.2)} \\
&= \lim_{r\to 0^+} \int_{Q\setminus B_r(P)} \alpha^*\rho^*(E(\Omega)) \\
&\quad \text{(as } \rho\alpha \text{ is the identity map on } Q\text{)} \\
&= -\lim_{r\to 0^+} \int_{Q\setminus B_r(P)} d\alpha^*(\Psi) \\
&\quad \text{(as } \rho^* E(\Omega) = -d\Psi \text{ on } STQ\text{)} \\
&= -\lim_{r\to 0^+} \int_{\partial B_r(P)} \alpha^*(\Psi) - \int_M \alpha^*(\Psi) \\
&\quad \text{(by Stokes' Theorem)} \\
&= \operatorname{ind}(X) - \int_M \alpha^*(\Psi),
\end{aligned}$$

so that

$$\operatorname{ind}(X) = \chi'_{orb}(Q) + \int_M \alpha^*(\Psi).$$

Similarly, if $n = 2m+1$ is odd, then as $E(\Omega) = 0$,

$$\begin{aligned} 0 &= \lim_{r\to 0^+} \int\limits_{Q\setminus B_r(P)} \alpha^*\rho^*(E(\Omega)) \\ &= -\lim_{r\to 0^+} \int\limits_{Q\setminus B_r(P)} d\alpha^*(\Psi) \\ &= -\lim_{r\to 0^+} \int\limits_{\partial B_r(P)} \alpha^*(\Psi) - \int\limits_M \alpha^*(\Psi) \\ &= \mathrm{ind}(X) - \int\limits_M \alpha^*(\Psi), \end{aligned}$$

so that

$$\mathrm{ind}(X) = \int\limits_M \alpha^*(\Psi).$$

In summary, we state the following.

Proposition 3.2.4 Let Q be a compact orbifold of dimension n with boundary M. Let X be vector field on Q which has a finite number of singularities, all of which occuring on the interior of Q. Then

$$\mathrm{ind}(X) = \begin{cases} \chi'_{orb}(Q) + \int\limits_M \alpha^*(\Psi), & n = 2m, \\ \int\limits_M \alpha^*(\Psi), & n = 2m+1. \end{cases}$$

3.3 The Thom Isomorphism Theorem for Orbifolds

In Section 3.4 below, we will show that, as is the case with manifolds, the cohomology class of the form Ψ is an invariant of $STQ_{|M}$, and hence does not depend on the various choices made. In order to characterize the cohomology class of Ψ in $H^n(STQ_{|M})$, we need to determine its relationship with the Euler and Thom classes of the tangent bundle $TQ_{|M}$. To this end, we develop the Thom Isomorphism Theorem for orbifolds in de Rham cohomology (Theorem 3.3.8).

3.3.1 The Thom Isomorphism in de Rham Cohomology

We begin by stating the following Theorem (taken from [14]). In this case, E_0 denotes the set of nonzero vectors in the vector bundle ξ with total space E (so that $E_0 := E\backslash\{\text{zero section}\}$), and F is a typical fiber (with $F_0 := F \cap E_0$, the set of nonzero vectors in F).

Theorem 3.3.1 (Thom Isomorphism Theorem) Let ξ be an oriented l-plane bundle with total space E. Then the cohomology group $H^i(E, E_0; \mathbb{Z})$ is zero for $i < l$, and $H^l(E, E_0; \mathbb{Z})$ contains one and only one cohomology class u whose restriction

$$u \mid (F, F_0) \in H^l(F, F_0; \mathbb{Z})$$

is equal to the preferred generator u_F for every fiber F of ξ. Furthermore, the correspondence $y \mapsto y \cup u$ maps $H^k(E; \mathbb{Z})$ isomorphically onto $H^{k+l}(E, E_0; \mathbb{Z})$.

We will be using the Thom isomorphism below in de Rham cohomology, so it will serve us to re-state the theorem in this context. The construction below is based on the work of Schwerdtfeger [21].

Let M be a manifold with E an l-dimensional vector bundle over M. Assuming a Euclidean metric on the bundle E, let B denote the unit disk bundle of E (i.e. the set of vectors v with $\|v\| \leq 1$) and $\partial B = S$ the unit sphere bundle. We let ρ denote the projection $\rho : E \to M$, as well as its restriction to B, S, etc. Let $B_0 = E_0 \cap B$, and let $\phi : B_0 \to S$ be the map $v \mapsto \frac{1}{\|v\|}v$.

The cohomology of the pair (E, E_0), then, is studied via forms on B relative to S (i.e. forms on B which vanish on the boundary). We state the following theorem from [21], noting that we have changed his notation and sign convention in a consistent and suggestive manner.

Theorem 3.3.2 (Schwerdtfeger) Let $E(\Omega) \in \Omega^l(M)$, $\Psi \in \Omega^{l-1}(S)$ be forms which

satisfy

$$dE(\Omega) = 0, \;\; d\Psi = -\rho^*(E(\Omega)), \;\; \int\limits_{S_p} \Psi = 1$$

where S_p denotes the fiber of S over an arbitrary point $p \in M$. Let $h : B \to \mathbb{R}$ be smooth with

$$h = \begin{cases} 0 & \text{inside of a } \delta\text{-neighborhood of the zero section} \\ \\ 1 & \text{outside of an } \epsilon\text{-neighborhood of the zero section} \end{cases}$$

where $0 < \delta < \epsilon < 1$.

Then the form

$$\tau = \rho^*(E(\Omega)) + d(h \cdot \rho^*(\Psi)) \in \Omega^l(B, S)$$

is a representative of the Thom class.

Quite clearly, in the case of a manifold, the forms $E(\Omega)$ and Ψ, as defined above, satisfy these conditions, and hence we may take this to be a definition of the Thom class for E (here again, we take the restriction of Ψ to the sphere bundle of E). Note that, restricted to S,

$$\tau = (\rho^* E(\Omega))_{|S} + d(\Psi_{|S}) = 0,$$

so that τ vanishes on the boundary. Note further that on M, $\tau = E(\Omega)$.

With this, we have that the map

$$\begin{array}{cccc} \psi : & \Omega^k(M) & \to & \Omega^{k+l}(B, S) \\ \\ : & \omega & \mapsto & \rho^*(\omega) \wedge \tau \end{array}$$

induces an isomorphism

$$H^k(M) \cong H^{k+l}(E, E_0),$$

where we are using the natural isomorphisms $H^{k+l}(E, E_0) \cong H^{k+l}(B, S)$ and $H^k(E) \cong H^k(M)$ induced in both cases by the inclusion of the latter into the former.

3.3.2 The Case of a Global Quotient

Let $Q := M/G$ be an n-dimensional, oriented, closed global quotient orbifold so that M is oriented, compact, and smooth and G is finite, and let E be a G-bundle of rank l on M equipped with a Euclidean metric. Then E/G naturally has the structure of an orbifold vector bundle over Q. Say that E/G carries a Euclidean metric (which is precisely a G-invariant metric on E). Recall that a differential form ω on Q is precisely a G-invariant differential form on M. Hence, if $(\Omega^k(M))^G$ denotes the G-invariant k-forms on M, then the quotient projection

$$\pi : M \to M/G = Q$$

induces

$$\pi^* : \Omega^k(Q) \to (\Omega^k(M))^G,$$

which is clearly an isomorphism of linear spaces. If we extend π to $E \to E/G$, then we have a similiar identification

$$\Omega^k(E/G) \cong (\Omega^k(E))^G.$$

Throughout this section, let π denote the projection of M onto Q (and its various extensions to bundles on M and their corresponding orbifold bundles on these Q). With E and E/G the bundles over M and Q, respectively (with respective projections ρ_M and ρ_Q), we let E_0 and $(E/G)_0$ denote the collection of nonzero vectors in each of these spaces, B_M and B_Q the ball bundles, S_M and S_Q the sphere bundles, etc.

Let $u_M \in H^n(E, E_0; \mathbb{R})$ denote the Thom class of the bundle E over M (tensoring the cohomology group with $\mathbb{R}$), and let $\tau \in \Omega^l(B_M, S_M)$ be the differential form given by Theorem 3.3.2 which represents the Thom class in de Rham cohomology. In general, if $\omega \in (\Omega^k(M))^G$ is a G-invariant differential form on M, we will identify it with the associated differential form on Q. We denote by $[\omega]_M$ its class in $H^k(M; \mathbb{R})$ and $[\omega]_Q$ its class in $H^k(Q; \mathbb{R})$ (and respectively, on E, (E, E_0), etc.). So in this notation, $u_M = [\tau]_M$.

We have that the map

$$\begin{array}{rccc} \psi: & H^k(M;\mathbb{R}) & \to & H^{k+l}(E,E_0;\mathbb{R}) \\ \\ : & [\omega]_M & \mapsto & [\rho_M^*(\omega)\wedge\tau_M]_M \end{array}$$

is an isomorphism (again useing the canonical isomorphisms $H^k(M;\mathbb{R}) \cong H^k(E;\mathbb{R})$ and $H^{k+l}(E,E_0;\mathbb{R}) \cong H^{k+l}(B_M,S_M;\mathbb{R})$).

Note that τ is G-invariant, as it is defined in terms of the forms Ψ and $E(\Omega)$, which are G-invariant whenever the metric is, and the function h can clearly be chosen to be G-invariant. Hence, $\tau \in \Omega^l(B_Q,S_Q)$ (via its identification with $(\Omega^l(B,S))^G$), and as $d\tau = 0$ clearly, τ represents a cohomology class in $H^l(Q;\mathbb{R})$.

Consider the map $\hat{\psi} : H^k(Q;\mathbb{R}) \to H^{k+l}(E/G,(E/G)_0;\mathbb{R})$, where $\hat{\psi}([\omega]_Q) = [\rho_Q^*(\omega)\wedge\tau]_Q$. We will represent this map using the isomorphisms $\Omega^k(E/G) \cong (\Omega^k(E))^G$ and $\Omega^k(B_Q,S_Q) \cong (\Omega^k(B_M,S_M))^G$ given above, and the obvious isomorphism

$$H^{k+l}(E/G,(E/G)_0;\mathbb{R}) \cong H^{k+l}(B_Q,S_Q;\mathbb{R}).$$

Hence, this map can be expressed on forms as $\hat{\psi}(\omega) = \rho_M^*(\omega)\wedge\tau$ for $\omega \in (\Omega^k(M))^G$.

Before we deal with $\hat{\psi}$, however, we need a lemma which will help us relate $\Omega^k(M)$ and $(\Omega^k(M))^G$. Essentially, it claims that if a G-invariant form is exact in $\Omega^k(M)$, then it is exact in $(\Omega^k(M))^G$.

Lemma 3.3.3 Suppose that $\eta_1 \in \Omega^{k-1}(M)$ with $d\eta_1 \in (\Omega^k(M))^G$. Then there is an $\eta_2 \in (\Omega^{k-1}(M))^G$ with $d\eta_1 = d\eta_2$.

Proof:

Using the averaging map, set

$$\eta_2 := \frac{1}{|G|}\sum_{\gamma\in G}\gamma^*\eta_1.$$

Then

$$\begin{aligned}
d\eta_2 &= d\left(\frac{1}{|G|}\sum_{\gamma\in G}\gamma^*\eta_1\right) \\
&= \frac{1}{|G|}\sum_{\gamma\in G}\gamma^* d\eta_1 \\
&= \frac{1}{|G|}\sum_{\gamma\in G} d\eta_1 \\
&\quad \text{(as } d\eta_1 \text{ is } G\text{-invariant)} \\
&= d\eta_1.
\end{aligned}$$

Moreover, for each $\gamma_0 \in G$,

$$\begin{aligned}
\gamma_0^*\eta_2 &= \gamma_0^*\left(\frac{1}{|G|}\sum_{\gamma\in G}\gamma^* d\eta_1\right) \\
&= \frac{1}{|G|}\sum_{\gamma\in G}\gamma_0^*\gamma^* d\eta_1 \\
&= \frac{1}{|G|}\sum_{\gamma\in G}(\gamma\gamma_0)^* d\eta_1 \\
&= \frac{1}{|G|}\sum_{\gamma\in G}\gamma^* d\eta_1 \\
&\quad \text{(as multiplication by an element simply permutes} \\
&\quad \text{the elements of a group)} \\
&= \eta_2,
\end{aligned}$$

so that η_2 is G-invariant.

Q.E.D.

Claim 3.3.4 The map $\hat{\psi} : H^k(Q;\mathbb{R}) \to H^{k+l}(E/G,(E/G)_0;\mathbb{R})$ defined above is injective.

Proof:

Suppose that $\omega_1, \omega_2 \in (\Omega^k(M))^G$ represent classes in $H^k(E/G;\mathbb{R})$ such that $\hat{\psi}([\omega_1]_Q) = \hat{\psi}([\omega_2]_Q)$. Hence, $[\rho_Q^*(\omega_1) \wedge \tau]_Q = [\rho_Q^*(\omega_2) \wedge \tau]_Q$.

As ω_1 and ω_2 are closed, they represent classes $[\omega_1]_M, [\omega_2]_M \in H^k(M;\mathbb{R})$. So as ψ is known to be an isomorphism here, and as $\psi([\omega_1]_M) = \psi([\omega_2]_M)$, we have that $[\omega_1]_M = [\omega_2]_M$. So there is an $\eta \in \Omega^{k-1}(M)$ such that

$$\omega_1 = \omega_2 + d\eta.$$

Note that, as

$$d\eta = \omega_1 - \omega_2,$$

and the ω_i are G-invariant, $d\eta$ is G-invariant. Applying Lemma 3.3.3, we can take η to be G-invariant, and hence $[\omega_1]_Q = [\omega_2]_Q$. So $\hat{\psi}$ is injective.

Q.E.D.

Claim 3.3.5 The map $\hat{\psi} : H^k(Q;\mathbb{R}) \to H^{k+l}(E/G,(E/G)_0;\mathbb{R})$ is surjective.

Proof:

Let $\zeta \in (\Omega^{k+l}(E,E_0))^G$ be a closed form. Then ζ represents a class $[\zeta]_M \in H^{k+l}(E,E_0;\mathbb{R})$, so that as ψ is an isomorphism, there is a closed $\omega \in \Omega^k(M)$ and an $\eta \in \Omega^{k-1}(M)$ such that $\zeta = \rho_M^*(\omega) \wedge \tau + d\eta$. As ω is closed, $d\omega = 0$ is G-invariant, so that ω can be taken to be G-invariant. So ω represents a class in $H^k(Q;\mathbb{R})$. Moreover, $d\eta = \zeta - \rho_M^*(\omega) \wedge \tau$ is G-invariant, so that η can be taken to be, Therefore, ζ is cohomologous to $\rho_M^*(\omega) \wedge \tau$ in $H^{n+l}(Q;\mathbb{R})$.

Q.E.D.

Claim 3.3.6 The cohomology class $[\tau]_Q \in H^l(E/G,(E/G)_0)$ does not depend on the connection.

Proof:

We need only show that the map $\pi^* : H^l(E/G,(E/G)_0) \to H^l(E,E_0)$ induced by the quotient projection $\pi : E \to E/G$ is injective. for the cohomology class of $\pi^*\tau$ is unique by Theorem 3.3.1. So suppose $\pi^*\omega_1 - \pi^*\omega_2 = d\eta$ for closed $\omega_1, \omega_2 \in \Omega^k(B_Q, S_Q)$ and some $\eta \in \Omega^{k-1}(B_M, S_M)$. Then as $d\eta = \pi^*(\omega_1 - \omega_2)$ is G-invariant, η can be chosen to be G-invariant, so that η is in the image of π^*. Hence $\omega_1 - \omega_2$ is exact.

Q.E.D.

Hence, we see that τ represents a unique cohomology class $u_Q \in H^l(E/G,(E/G)_0;\mathbb{R})$ which induces an isomorphism $\rho_Q^*(\cdot) \cup u_Q : H^k(Q;\mathbb{R}) \to H^{k+l}(E/G,(E/G)_0;\mathbb{R})$.

In summary, we state:

Proposition 3.3.7 (The Thom Isomorphism for Global Quotients) Let $Q = M/G$ be an n-dimensional, oriented, closed global quotient orbifold so that M is oriented, compact, and smooth and G is finite, and let E be a G-bundle on M. Then, with the form τ as defined above, the map

$$\begin{array}{rccc} \hat{\psi}: & H^k(Q;\mathbb{R}) & \to & H^{k+l}(E/G,(E/G)_0;\mathbb{R}) \\ \\ : & [\omega]_Q & \mapsto & [\rho_Q^*(\omega) \wedge \tau]_Q \end{array}$$

is an isomorphism. Hence, if $u_Q = [\tau]_Q$ denotes the cohomology class in $H^l(E/G,(E/G)_0;\mathbb{R})$ represented by τ, then the map $\rho_Q^*(\cdot) \cup u_Q : H^k(Q;\mathbb{R}) \to H^{k+n}(E/G,(E/G)_0;\mathbb{R})$ is an isomorphism. Moreover, u_Q does not depend on the metric or connection.

In the next section, we will apply these results locally to a bundle E over a general orbifold by considering each set U with chart $\{V, G, \pi\}$ to be a global quotient V/G,

and the bundle E to be the quotient $(\pi^*E)/G$. However, in these cases, the set V is not compact. In order to justify such an application, we note that although Schwerdtfeger's results only apply to compact manifolds, Theorem 3.3.1 applies in general, so that there is a Thom class in $H^k(\pi^*(E), (\pi^*(E))_0)$ with the desired properties. Moreover, as we can always take V to be a bounded open ball in $\mathbb{R}^n$ and extend the metric and connection to a larger ball in $\mathbb{R}^n$ which contains $\overline{V}$. Hence, as the Thom class can be characterized by a completely local property (that its restriction to each fiber F is the preferred generator of $H^k(F, F_0)$ with respect to the orientation), the class of τ must coincide with the Thom class on the interior of this ball. So in this case, the above construction is still valid.

3.3.3 The Case of a General Orbifold

For the case of a general closed orbifold Q with orbibundle $\rho : E \to Q$, we will use the fact that Q is locally a global quotient; i.e. that each point is contained in a neighborhood which is a global quotient. The proof of the Thom Isomorphism Theorem, then, will involve an induction following Milnor and Stasheff [14]. We note that for each open $U \subset Q$ that is uniformized by $\{V, G, \pi\}$, U is given the structure of a global quotient orbifold via $\tilde{\pi} : V/G \cong U$. Hence, applying the note in the last subsection, the Thom Isomorphism Theorem is known locally.

First, we let E carry a Euclidean metric. Then we may take the definition of the Thom class, τ, as given above, in terms of the global forms Ψ and $E(\Omega)$. As $d\tau = 0$, τ represents a cohomology class $[\tau] =: u \in H^n(B, S; \mathbb{R}) \cong H^n(E, E_0; \mathbb{R})$, which we define to be the Thom class of Q (here, B and S are defined to be the ball bundle and sphere bundle, respectively, of E with respect to the metric, as in the last subsection). Note that, for each uniformized $U \subset Q$, we may give U the restricted metric of Q, and then $\tau_{|U}$ is the Thom form of U. We now proceed with the induction.

Suppose $Q = U_1 \cup U_2$ is the union of two open sets, with U_i having chart

$\{V_i, G_i, \pi_i\}$ for $i = 1, 2$, such that each V_i is a bounded ball in $\mathbb{R}^n$, and each G_i acts linearly. Then applying the note above and the case for global quotients, the Thom Isomorphism holds for each $E_{|U_i}$. Note further that, as $W := U_1 \cap U_2 \subset U_1$, W is also given the structure of a quotient via the uniformization of U_1.

We have the following two Meier-Vietoris sequences (using coefficients in $\mathbb{R}$ throughout):

$$\cdots \to H^{k-1}(W) \to H^k(Q) \to H^k(U_1) \oplus H^k(U_2) \to H^k(W) \to \cdots$$

and

$$\cdots \to H^{k+l-1}(E_{|W}, (E_{|W})_0) \to H^{k+l}(E, E_0) \to$$

$$\to H^{k+l}(E_{|U_1}, (E_{|U_1})_0) \oplus H^{k+l}(E_{|U_2}, (E_{|U_2})_0) \to H^{k+l}(E_{|W}, (E_{|W})_0) \to \cdots$$

Using the fact that the isomorphism is known for all but one step in this sequence, we have the following for each k,

$$\begin{array}{ccccc} H^{k-1}(W) & \to & H^k(Q) & \to & \\ \downarrow\cong & & \downarrow * & & \\ H^{k+l-1}(E_{|W}, (E_{|W})_0) & \to & H^{k+l}(E, E_0) & \to & \end{array}$$

$$\begin{array}{ccccc} \to & H^k(U_1) \oplus H^k(U_2) & \to & H^k(W) \\ & \downarrow\cong & & \downarrow\cong \\ \to & H^{k+l}(E_{|U_1}, (E_{|U_1})_0) \oplus H^{k+l}(E_{|U_2}, (E_{|U_2})_0) & \to & H^{k+l}(E_{|W}, (E_{|W})_0) \end{array}$$

Each of the vertical isomorphisms is given by $\rho^*(\cdot) \wedge \tau$ on the level of forms; similarly, by $\rho^*(\cdot) \cup u$ in cohomology (with τ and u appropriately restricted), so that the diagram clearly commutes. Applying the Five Lemma gives us that $(*)$ is also an isomorphism,

and that the isomorphism is given by $\rho^*(\cdot) \wedge \tau$. Hence, we have shown the Thom isomorphism in this case.

Now, suppose Q is any closed orbifold with orbifold vector bundle E of rank l, and let $\{U_i\}_{i=1}^k$ be a cover of Q such that each U_i is uniformized by $\{V_i, G_i, \pi_i\}$. If $k = 1$, then Q is a global quotient, and Thom isomorphism is known. For $k > 1$, assuming as our inductive hypothesis that the Thom isomorphism holds for $E_{|U_1 \cup \cdots \cup U_{k-1}}$, applying the above argument to the two sets $U_1 \cup \cdots \cup U_{k-1}$ and U_k shows that it holds for E.

With this, we have proven the following.

Theorem 3.3.8 (The Thom Isomorphism for a Closed Orbifold) Let Q be a closed oriented orbifold and $\rho : E \to Q$ an oriented orbifold vector bundle of rank l over Q. Let τ be defined as in Theorem 3.3.2, and then the map

$$\begin{array}{rccc} \hat{\psi} : & H^k(Q;\mathbb{R}) & \to & H^{k+l}(E, E_0;\mathbb{R}) \\ \\ : & [\omega]_Q & \mapsto & [\rho_Q^*(\omega) \wedge \tau]_Q \end{array}$$

is an isomorphism. Moreover, with respect to each chart $\{V, G, \pi\}$, $\pi^*\tau$ restricts to the preferred generator of $H^l(F, F_0\ \mathbb{Z})$ (using the isomorphism between $H^l(F, F_0\ \mathbb{R})$ and $H^l(F, F_0\ \mathbb{Z}) \otimes \mathbb{R}$). In particular, the class of τ does not depend on the metric or connection.

Note that, by its construction, it is trivial that the restriction of τ to Q is $E(\Omega)$ (and hence the restriction of u to Q is the cohomology class of $E(\Omega)$ in $H^l(Q)$). Therefore, $E(\Omega)$ is closed, and cohomology class it represents in $H^l(Q)$ does not depend on the metric. We refer to this class as $e(E)$, the **Euler class of E as an orbifold bundle**. In the case $E = TQ$, $e(TQ) =: e(Q)$ is simply the **Euler class of Q as an orbifold.** In general it does not coincide with the Euler class of the underlying space of Q, but is a rational multiple thereof.

3.4 The Theorems on the Level of Cohomology

3.4.1 Invariance of the Integrands on the Metric

We return to the case where Q has boundary, $M = \partial Q$, and $E = TQ$. We have the formula (Proposition 3.2.4)

$$\operatorname{ind}(X) = \begin{cases} \chi'_{orb}(Q) + \int\limits_M \alpha^*(\Psi), & n = 2m, \\ \\ \int\limits_M \alpha^*(\Psi), & n = 2m+1, \end{cases}$$

which generalizes the Poincaré-Hopf Theorem to the case of a compact orbifold with boundary. In this section, we characterize the cohomology class of Ψ in $H^n(STQ_M)$ in order to show that it does not depend on the metric. This section again follows [22].

Recall that $d\Psi = -\rho^* E(\Omega)$ on STQ. However, as $TQ_{|M}$ is isomorphic to $TM \oplus \nu$ where ν denotes the trivial bundle on M of rank 1, and hence that $\Omega_{i,j} = 0$ whenever $i > 1$ or $j > 1$, $E(\Omega)$ clearly vanishes on M. So Ψ is a closed form on $STQ_{|M}$, and hence represents a cohomology class $\Upsilon \in H^n(STQ_{|M}; \mathbb{R})$. We have seen that

$$\alpha_0^*(\Upsilon) = \frac{1}{2} e(TM),$$

where α_0 is the outward-pointing unit normal vector field on M and $e(TM)$ denotes the cohomology class of $E(\Omega_{|M})$ in $H^{n-1}(M; \mathbb{R})$. Moreover, the integral of Ψ on a fiber of $STQ_{|M}$ over a point p is $\frac{1}{|I_p|}$. Thus, the following is clear.

Claim 3.4.1 Let $p \in U_p \subset M$ where U_p is uniformized by $\{V_p, G_p, \pi_p\}$. If $\iota : S^n \to \pi_p^* STQ_{|M}$ is an orientation-preserving isometry from S^n to a fiber of the sphere bundle of $\pi_p^* TQ$ (i.e. the sphere bundle STQ pulled back over V), then then $\iota^* \pi_p^*(\Upsilon) = \frac{1}{|G_p|} s^n$, where s^n denotes the canonical generator of $H^n(S_n; \mathbb{Z})$.

Note that in the case where $G = 1$, we can identify U with V via π_p, so that $\iota^* \pi_p^* \Upsilon = \iota^* \Upsilon$.

With this, we may characterize the class Υ by developing the Gysin sequence of the tangent bundle. Let E now denote the restriction $TQ_{|M}$ and E_0 the nonzero vectors of E. We follow Milnor and Stasheff [14] and Sha [22]. All coefficient groups are understood to be $\mathbb{R}$.

To simplify the notation, we let E denote $TQ_{|M}$. We have the cohomology exact sequence

$$\cdots \longrightarrow H^j(E, E_0) \longrightarrow H^j(E) \longrightarrow H^j(E_0) \longrightarrow H^{j+1}(E, E_0) \longrightarrow \cdots$$

We may replace $H^j(E)$ with $H^j(M)$ using the natural isomorphism induced by ρ. Similarly, applying the Thom Isomorphism, we may replace $H^j(E, E_0)$ with $H^{j-n-1}(M)$. The restriction map $H^j(E, E_0) \to H^j(E)$ composed with $\cdot \cup u$ results in $\cdot \cup e$ (where e is the Euler class of the orbifold bundle E):

$$\cdots \longrightarrow H^{j-n-1}(M) \xrightarrow{\cup e} H^j(M) \longrightarrow H^j(E_0) \longrightarrow H^{j-n}(M) \xrightarrow{\cup e} \cdots$$

Now $H^j(E_0)$ is canonically isomorphic to $H^j(SE)$, so that again using ρ for the restricted projection $\rho : SE \to M$ we obtain

$$\cdots \longrightarrow H^{j-n-1}(M) \xrightarrow{\cup e} H^j(M) \xrightarrow{\rho^*} H^j(SE) \longrightarrow H^{j-n}(M) \xrightarrow{\cup e} \cdots$$

However, we have noted that the Euler class e of E is zero, so that setting $j = n$, we obtain

$$0 \longrightarrow H^n(M) \longrightarrow H^n(M) \xrightarrow{\rho^*} H^n(STQ) \longrightarrow H^0(M) \xrightarrow{\cup e} 0$$

Finally, we choose $\iota : S^n \to SE_p$ to be an isometry (as above) to a fiber over a point p with trivial isotropy, so that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n(M) & \xrightarrow{\rho^*} & H^n(SE_{|M}) & \rightarrow & H^0(M) & \xrightarrow{\cup e} & 0 \\ & & & & \downarrow \iota^* & & \downarrow\cong & & \\ & & & & H^n(S^n) & \xrightarrow{\cong} & H^0(\text{point}) & & \end{array}$$

gives the split exact sequence

$$0 \longrightarrow H^n(M) \xrightarrow{\rho^*} H^n(SE) \xrightarrow{\iota^*} H^n(S^n) \longrightarrow 0$$

With this, as α_0^* is a left inverse of ρ^* (recall that $\alpha_0 : M \to SE$ is the section induced by the outward unit normal vector field on M), we have $H^n(SE) = \rho^*(H^n(M)) \oplus (\alpha_0^*)^{-1}(0)$. With respect to this decomposition, based on the properties of Ψ, Υ factors into $\frac{1}{2}\rho^*(e) + \iota^{-1}_{|(\alpha_0^*)^{-1}(0)}(s^n)$. Hence, this characterizes Υ, and in particular shows that it does not depend on the choices made in the definition of Ψ. Note that choosing ι to map to the fiber over a singular point p will introduce a coefficient of $\frac{1}{|G_p|}$ in the above expression.

With this, Proposition 3.2.4 becomes

Theorem 3.4.2 (The First Poincaré-Hopf Theorem for Orbifolds with Boundary) Let Q be a compact orbifold with boundary M. Let X be vector field on Q which has a finite number of singularities, all of which occuring on the interior of Q. Then

$$\operatorname{ind}(X) = \begin{cases} \chi'_{orb}(Q) + \alpha^*\Upsilon([M]), & n = 2m, \\ \\ \alpha^*\Upsilon([M]), & n = 2m+1. \end{cases}$$

Chapter 4

The Gauss-Bonnet Integrand in Chen-Ruan Cohomology

4.1 Introduction

The goal of this chapter is to examine the results of the previous chapter in terms of the orbifold cohomology developed in [3]. In particular, we are interested in cohomology classes corresponding to those of $E(\Omega)$ and Ψ. Roughly speaking, the Chen-Ruan orbifold cohomology of an orbifold Q contains the usual cohomology of Q as a direct summand, but contains as well the cohomology groups of the twisted sectors, corresponding to irreducible components of the singular set of Q. With respect to this decomposition, the new characteristic classes will project to the usual ones. They will, however, have additional lower-degree terms, which correspond to the contributions of the singular sets. Our results, then, will involve the Euler characteristic of the underlying topological space instead of the orbifold Euler characteristic. Taking the point of view that an orbifold structure is a generalization of a differentiable structure on a manifold, we obtain results for orbifolds much more in keeping with the original Gauss-Bonnet and Poincaré-Hopf Theorems.

Throughout this chapter, orbifolds and orbifolds with boundary will be taken to admit almost complex structures.

4.2 Chen-Ruan Orbifold Cohomology

In this chapter, we will be working with the orbifold cohomology theory developed by [3]. We will not develop this cohomology theory here, but will collect a summary for the sake of making the notation explicit. For the most part, we follow the notation in [3], [17], and [18].

Let Q be an orbifold, and select for each $p \in Q$ a chart $\{V_p, G_p, \pi_p\}$ at p. Then the set

$$\tilde{Q} = \{(p, (g)_{G_p}) : p \in Q, g \in G_p\}$$

(where $(g)_{G_p}$ is the conjugacy class of g in G_p) is naturally an orbifold, with local charts

$$\{\pi_{p,g} : (V_p^g, C(g)) \to V_p^g / C(g) : p \in Q, g \in G_p\},$$

where V_p^g is the fixed point set of g in V_p and $C(g)$ is the centralizer of g in G_p. If Q is closed, then $\tilde{Q}$ is closed, but it need not be connected, and its connected components need not be of the same dimension. An equivalence relation can be placed on the elements of the groups G_p so that if T denotes the set of equivalence classes and (g) the equivalence class of g,

$$\tilde{Q} = \bigsqcup_{(g) \in T} \tilde{Q}_{(g)}$$

where

$$\tilde{Q}_{(g)} = \{(p, (g')_{G_p}) : g' \in G_p, (g')_{G_p} \in (g)\}.$$

The map $\pi : \tilde{Q} \to Q$ with $(p, (g)_{G_p}) \mapsto p$, is a C^∞ map.

If Q is an almost complex orbifold, a function $\iota : \tilde{Q} \to \mathbb{Q}$ is defined which is constant on the connected components of $\tilde{Q}$. If $n_{(g)}$ denotes the codimension of $\tilde{Q}_{(g)}$ in Q, then $2\iota_{(g)} \leq n_g$, with equality only when $g = 1$. This is called the degree shifting number of (g). The orbifold cohomology groups are defined by

$$H^d_{orb}(Q) = \bigoplus_{(g) \in T} H^{d - 2\iota_{(g)}}(\tilde{Q}_{(g)}),$$

where the groups on the right side are the usual de Rham cohomology groups of the orbifolds $\tilde{Q}_{(g)}$.

Since each $\tilde{Q}_{(g)}$ can be realized as a subset of Q, geometric constructions (i.e. bundles and their sections) on Q can be naturally extended to geometric constructions on $\tilde{Q}$. In what follows, we wish to extend the characteristic classes of bundles over Q to characteristic classes of associated bundles over $\tilde{Q}$; however, pulling back such bundles via π will be insufficient. In particular, if $E \to Q$ is a rank k orbibundle and $p \in Q$ is a singular point contained in a singular set of dimension $l < k$, then the maximal vector space in a fiber over p has dimension l. This implies that any k-form on Q is zero on p (recall that it is required of sections s of orbibundles that for each $q \in Q$, $s(q)$ is contained in the subspace E^q of the fiber over q which is fixed by G_q). In particular, any form representing the Euler class of E is zero at p, so that the pull-back π^* of this form will be zero on each connected component $\tilde{Q}_{(g)}$ of $\tilde{Q}$ of dimension is less than k. This is particularly disappointing in the case of the tangent bundle, in that no twisted sectors make contributions to the Euler class.

Instead, we will associate to each bundle $E \to Q$ a bundle $\tilde{E} \to \tilde{Q}$ whose dimension on each component $\tilde{Q}_{(g)}$ is equal to $k - n + l$, where k is the rank of E, n is the dimension of Q, and l is the dimension of $\tilde{Q}_{(g)}$ (i.e. the rank of E minus the codimension of $\tilde{Q}_{(g)}$ in Q). We will then apply the Chern-Weil construction to a connection on $\tilde{E}$ in order to define characteristic classes in $H^*_{orb}(Q)$ which are invariants of E.

4.3 Chen-Ruan Orbifold Cohomology for Orbifolds with Boundary

In this section, we generalize orbifold cohomology to the case of orbifolds with boundary. This is a straightforward generalization following [3].

Let Q be an n-dimensional orbifold with boundary M. Again, we let

$$\tilde{Q} = \{(p, (g)_{G_p}) : p \in Q, g \in G_p\}.$$

Then we have:

Lemma 4.3.1 The set $\tilde{Q}$ is naturally an orbifold with boundary, with projections given by

$$\{\pi_{p,g} : (V_p^g, C(g)) \to V_p^g/C(g) : p \in Q, g \in G_p\},$$

for each chart (V_p, G_p, π_p) at $p \in Q$. Here, V_p^g denotes the fixed point set of g in V_p and $C(g)$ is the centralizer of g in G_p. If Q is compact, then so is $\tilde{Q}$. The map $\pi : \tilde{Q} \to Q$ is a $\mathcal{C}^\infty$ map.

For the proof of this lemma for the case that Q does not have boundary, see [3]. The proof for the case with boundaries is identical.

Lemma 4.3.2 Let Q be an orbifold with boundary M. Then $\partial\tilde{Q} = \tilde{M}$.

Proof:

Let $(p, (g)_{G_p})$ be a point in $\partial\tilde{Q}$. Then $(p, (g)_{G_p})$ is contained in a chart of the form $\{V_p^g, C(g), \pi_{p,g}\}$, induced by a chart $\{V_p, G_p, \pi_p\}$ for Q at p. As $(p, (g)_{G_p})$ is in th boundary of $\tilde{Q}$, V_p^g is diffeomorphic to $\mathbb{R}_+^k$ for some k. However, as $G_p < O(n)$, its fixed-point set in $\mathbb{R}^n \supset V_p$ is a subspace, so that V_p must be diffeomorphic to $\mathbb{R}_+^n$ (where n is the dimension of Q). Therefore, $p \in M$.

Conversely, suppose $p \in M$ is a point in the boundary of Q, and let $\{V_p, G_p, \pi_p\}$ be a chart at p. Then $V_p \cong \mathbb{R}_+^n$, and any lift $\tilde{p}$ of p into V_p is contained in ∂V_p. For any $g \in G_p$, the element $(p, (g)_{G_p})$ of $\tilde{Q}$ is covered by the chart $\{V_p^g, C(g), \pi_{p,g}\}$, and any lift of $(p, (g)_{G_p})$ into V_p^g is clearly an element of ∂V_p^g. Therefore, $(p, (g)_{G_p})$ represents a point in $\partial\tilde{Q}$. With this, we note that any point $(p, (g)_{G_p}) \in \tilde{M}$ arises in such a way, and is contained in a chart for $\tilde{M}$ induced by a chart for $\tilde{Q}$ (and hence by a chart for Q).

Q.E.D.

We review the description of the connected components of $\tilde{Q}$, treating the case that Q has boundary.

Let $\{V_p, G_p, \pi_p\}$ be an orbifold chart for Q at a point $p \in Q$, and let $q \in U_p = \pi_p(V_p)$. Let $\{V_q, G_q, \pi_q\}$ be an orbifold chart at q with $U_q \subset U_p$, and then the definition of an orbifold gives us an injection $\lambda_{qp} : \{V_q, G_q, \pi_q\} \to \{V_p, G_p, \pi_p\}$. The injective homomorphism $f_{qp} : G_q \to G_p$ is well-defined up to conjugation, so that it defines for each conjugacy class $(g)_{G_q}$ a conjugacy class $(f_{qp}(g))_{G_p}$. We say that $(g)_{G_q} \sim (f_{qp}(g))_{G_p}$, which defines an equivalence relation on the elements of the local groups. Let (g) denote the equivalence class of a group element g; note that it is no longer important to state the particular local group from which g was taken. For each equivalence class, we let

$$\tilde{Q}_{(g)} := \{(p, (h)_{G_p}) | h \in G_p, (h) = (g)\},$$

and then $\tilde{Q} = \bigsqcup_{(g)\in T} \tilde{Q}_{(g)}$. In particular, following [3], we call $\tilde{Q}_{(1)}$ the **nontwisted sector** and each $\tilde{Q}_{(g)}$ for $g \neq 1$ a **twisted sector**. It is worth noting that in the case that $Q = M/G$ with M a manifold and G a finite group, the equivalence relation reduces to that of conjugation in G.

Example 4.3.3 For the case of a point $Q = \{p\}$ with the trivial action of a finite group G (see Example 2.1.8), the above defined equivalence relation reduces to conjugation within the group, and $\tilde{Q}$ consists of one point for each conjugacy class. For an element $g \in G$, the point $(p, (g)_G)$ corresponding to (g) has the trivial action of $C(g)$, the centralizer of $g \in G$.

Example 4.3.4 If Q is the $\mathbb{Z}_k$-teardrop (see Example 2.1.9), then $\tilde{Q}$ has k connected components. The nontwisted sector is diffeomorphic to Q, while each of the $k-1$ other components are points with the trivial action of $\mathbb{Z}_k$.

Example 4.3.5 For $Q = \mathbb{R}^3/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts via the antipodal map (see Example 2.1.10), $\tilde{Q}$ has two connected components, one diffeomorphic to Q and one given by a point with trivial $\mathbb{Z}_2$-action.

Example 4.3.6 Consider the case where Q is the $\mathbb{Z}_k$-$\mathbb{Z}_l$-solid hollow football (see Example 2.1.11) with k and l relatively prime. Then $\tilde{Q}$ has $k+l-1$ connected components. The component corresponding to the identity element (in both groups) is diffeomorphic to Q. Each of the $k+l-2$ other components is diffeomorphic to a closed interval $[0,1]$ with the trivial action of $\mathbb{Z}_k$ or $\mathbb{Z}_l$ (there are $k-1$ components with trivial $\mathbb{Z}_k$-action and $l-1$ components with trivial $\mathbb{Z}_l$-action).

If $k = l$, then the equivalence relation on conjugacy classes defines an isomorphism between the two $\mathbb{Z}_k$, and Q is a global quotient. Then there are exactly k connected components of $\tilde{Q}$: one, corresponding to the identity in $\mathbb{Z}_k$, is diffeomorphic to Q, while each of the others is given by $[-2,-1] \cup [1,2]$ with trivial $\mathbb{Z}_k$-action (these two pieces correspond to the fixed-point set of the nonidentity elements of $\mathbb{Z}_k$, which lie on the z-axis).

If k and l are not relatively prime, then letting $s := \gcd(k,l)$, Q can be expressed as a global quotient of the $\mathbb{Z}_{k/s}$-$\mathbb{Z}_{l/s}$-football by $\mathbb{Z}_s$, and then the structure of $\tilde{Q}$ can be determined using both of the above constructions.

Example 4.3.7 Suppose Q is an orbifold with boundary homeomorphic to the closed 3-disk whose singular set, located on the interior of the disk, is that given in Example 2.2.5. Then there are six equivalence classes in T, each containing exactly one element of $G_1 \cong D_6$. The nontwisted sector is again diffeomorphic to Q. The twisted sector corresponding to the equivalence class of R^x_π is given by S^1 with a trivial $\langle R^x_\pi \rangle \cong \mathbb{Z}_2$-action. The twisted sectors corresponding to $R^z_{\frac{2\pi}{3}}$ and $(R^z_{\frac{2\pi}{3}})^2$ are both given by S^1 with trivial $\langle R^z_{\frac{2\pi}{3}} \rangle \cong \mathbb{Z}_3$-action. The twisted sectors corresponding to the elements $(R^x_\pi)(R^z_{\frac{2\pi}{3}})$ and $(R^x_\pi)(R^z_{\frac{2\pi}{3}})^2$ are both points with trivial D_6-action.

Now, suppose Q admits an almost complex structure J. As in the case without boundary, $\tilde{Q}$ inherits an almost complex structure $\tilde{J}$ from Q (note that this almost structure is on $\pi^* TQ$, which is not generally the same as $T\tilde{Q}$. For each $p \in Q$ and

chart $\{V_p, G_p, \pi_p\}$ at p, the almost complex structure defines a homomorphism $\rho_p : G_p \to GL(n, \mathbb{C})$ where n is the dimension of Q over $\mathbb{C}$. For each $g \in G_p$, $\rho_p(g)$ can be expressed as

$$\begin{bmatrix} e^{2\pi i m_{1,g}/m_g} & 0 & \cdots & 0 \\ 0 & e^{2\pi i m_{2,g}/m_g} & & \vdots \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i m_{n,g}/m_g} \end{bmatrix}$$

with m_g the order of $\rho_p(g)$ and $0 \le m_{i,g} < m_g$. Following [3], define

$$\begin{array}{rccl} \iota & : & \tilde{Q} & \to \mathbb{Q} \\ & : & (p, (g)_{G_p}) & \mapsto \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}, \end{array}$$

and then ι defines a map which is constant on each $\tilde{Q}_{(g)}$. Let $\iota_{(g)} := \iota[(p, (g)_{G_p})]$ for any $(p, (g)_{G_p}) \in \tilde{Q}_{(g)}$. We refer to $\iota_{(g)}$ as the **degree shift number** of $\tilde{Q}_{(g)}$.

Definition 4.3.8 Let Q be an orbifold with boundary that admits an almost complex structure J. The **relative orbifold cohomology groups** are defined to be

$$H^d_{orb}(Q) = \bigoplus_{(g) \in T} H^{d-2\iota_{(g)}}(Q_{(g)}),$$

where $H^{d-2\iota_{(g)}}(Q_{(g)})$ is either the singular or de Rham relative cohomology group of $Q_{(g)}$.

In the sequel, we will examine analogs of the theorems of the previous chapter with the characteristic classes taken to be elements of these cohomology groups.

4.4 The Gauss-Bonnet and Poincaré-Hopf Theorems in Chen-Ruan Cohomology

We begin with a lemma.

Let Q be a closed orbifold of dimension n, and let $\rho : E \to Q$ be an orbifold vector bundle of rank k. Suppose further that E has the following property: for any chart $\{V_p, G_p, \pi_p\}$ at p, and any subgroup H of G_p, the codimension of the fixed point set of H in the fiber E_p is equal to the codimension of the fixed point set of H in V_p (or is zero if the codimension of the fixed point set in V_p is greater than the rank k of E). This is the case, for instance, for the tangent and cotangent bundles, their exterior powers, etc. (which can be verified using a metric and the exponential map). As E is an orbifold, we may apply the construction to form $\tilde{E}$.

Lemma 4.4.1 With E, Q as above, $\tilde{E}$ is naturally an orbifold vector bundle over $\tilde{Q}$, and the rank of $\tilde{E}$ on each connected component $\tilde{Q}_{(g)}$ of dimension l is $k - n + l$. In particular, the fiber of $\tilde{E}$ is zero on any $\tilde{Q}_{(g)}$ with codimension larger than k.

Note that the restriction on the group actions on E is stronger than the requirement that E be a so-called good orbifold vector bundle (see [17] for the definition; our definition of orbifold vector bundle coincides with Ruan's definition of a good orbifold vector bundle). However, we will be applying this result to the tangent bundle only. For general (good) orbifold vector bundles, $\tilde{E}$ is still naturally a vector bundle over $\tilde{Q}$, but the rank of $\tilde{E}$ over a connected component of $\tilde{Q}$ depends on the group action on E.

Proof:

First, we note that as the local groups and injections of E are precisely those of Q, the set T is identical for both orbifolds. Let $\tilde{\rho} : \tilde{E} \to \tilde{Q}$ be defined by $\tilde{\rho}(e, (g)) = (\rho(e), (g))$. Then $\tilde{\rho}$ is certainly well-defined, as e is fixed by a group element if $\rho(e)$ is.

Fix a point $(e, (g)) \in \tilde{E}$, where $e \in E$ and $(g) \in T$, and let $p := \rho(e) \in Q$ denote the projection of e. Then by the definition of $\tilde{E}$, $(e, (g))$ is contained in a uniformizing

set $\{V_e^g, C(g), \pi_{e,g}\}$ induced by a uniformizing set $\{V_e, C(g), \pi_{e,g}\}$ of e. Moreover, we can take this orbifold chart for E to be a uniformizing system $(V_p \times \mathbb{R}^k, G_p, \tilde{\pi}_p)$ of the rank k orbifold bundle induced by a system $\{V_p, G_p, \pi_p\}$ near p in Q. Hence, $V_e = V_p \times \mathbb{R}^k$, $G_e = G_p$, and $\pi_e = \tilde{\pi}_p$.

By our definition of orbifold vector bundle, the kernel of the G_p action on the $\mathbb{R}^k$ fiber over a point $y \in V_p$ is the kernel of the G_p action on y. Hence, the fixed point set V_e^g of g in $V_e = V_p \times \mathbb{R}^k$ is $V_p^g \times \mathbb{R}^{k-n+l}$ (or the zero vector space if $k - n + l \leq 0$), where l is the dimension of V_p^g (i.e. k minus the codimension of V_p^g in V_p). If we define the map $\tilde{\rho}_2 : V_e^g = V_p^g \times \mathbb{R}^{k-n+l} \to V_p^g$ by projection onto the first factor, then for any $f = (y, v) \in V_e^g$ (say $\pi_p(y) = q \in Q$), we have that

$$\begin{aligned}
\pi_{p,g} \circ \tilde{\rho}_2(f) &= \pi_{p,g}(y) \\
&= (q, (g)) \\
&= \tilde{\rho}((q, (g)), v) \\
&= \tilde{\rho} \circ \pi_{e,g}(f),
\end{aligned}$$

and hence that $\pi_{p,g} \circ \tilde{\rho}_2 = \tilde{\rho} \circ \pi_{e,g}$.

Hence,

$$\{V_e^g, C(g), \pi_{e,g}\} = \{V_p^g \times \mathbb{R}^{k-n+l}, C(g), \pi_{e,g}\}$$

is a rank $k - n + l$ uniformizing system for a bundle over the uniformizing system $\{V_p^g, C(g), \pi_{p,g}\}$ about $(p, (g)) \in \tilde{Q}$.

We have that any point $(e, (g))$ of $\tilde{E}$ is contained in a bundle uniformizing system over a uniformizing system of $(\rho(e), (g)) \in \tilde{Q}$ with projection $\tilde{\rho}$. Given a compatible cover $\mathcal{U}$ of $\tilde{Q}$ (which can be taken to be induced by a compatible cover of Q) an injection $\tilde{\lambda} : \{V_p^g, C(g), \pi_{p,g}\} \to \{V_q^h, C(h), \pi_{q,h}\}$ is always the restriction of an injection λ :

$\{V_p, G_p, \pi_p\} \to \{V_q, G_q, \pi_q\}$ (by [20], Lemma 1, there is a bijection between elements of G_q and such injections; hence, restricting to the subgroup $C(h)$ decreases the number of injections). Then the transition map $\tilde{\phi} : V_p^g \to Aut(\mathbb{R}^{k-n+l})$ is simply the restriction of the corresponding transition map $\phi : V_p \to Aut(\mathbb{R}^n)$ for E to the fixed-point set V_p^g. Therefore, these uniformizing systems patch together to give $\tilde{E}$ the structure of an orbibundle over $\tilde{Q}$.

Q.E.D.

Applying the above argument to the tangent bundle shows that $T(\tilde{Q}) = \widetilde{TQ}$; i.e. the tangent bundle of $\tilde{Q}$ is the collection of twisted sectors of the tangent bundle of Q. Similarly, the constructions of the cotangent, exterior power, and tensor bundles commute with this construction. Moreover, any smooth section ω of the bundle E naturally induces a smooth section $\tilde{\omega}$ of the bundle $\tilde{E}$ via $\tilde{\omega} : (p, (g)) \mapsto (\omega(p), (g))$.

Theorem 4.4.2 (The Second Gauss-Bonnet Theorem for Closed Orbifolds) Let Q be a closed oriented orbifold of dimension n, and suppose Q carries a connection ω with curvature Ω. Let $\tilde{\omega}$ denote the induced connection ω on $\widetilde{TQ}$, and let $\tilde{\Omega}$ denote its curvature. Let $E(\tilde{\Omega})$ denote the Euler curvature form of $\tilde{\Omega}$, defined on $\tilde{Q}$, and then

$$\int_{\tilde{Q}} E(\tilde{\Omega}) = \chi(Q),$$

the Euler characteristic of the underlying topological space of Q. Moreover, if Q is almost complex, then $E(\tilde{\Omega})$ represents an element of the cohomology ring $H^*_{orb}(Q)$ which is independent of the connection on Q.

Proof:

We first note that on each $\tilde{Q}_{(g)}$, $E(\tilde{\Omega})$ is a representative of the Euler class of

$\tilde{Q}_{(g)}$, so that, by the Gauss-Bonnet theorem for orbifolds,

$$\begin{aligned} \int_{\tilde{Q}} E(\tilde{\Omega}) &= \sum_{(g)\in T} \int_{\tilde{Q}_{(g)}} E(\tilde{\Omega}) \\ &= \sum_{(g)\in T} \chi_{orb}(\tilde{Q}_{(g)}), \end{aligned}$$

where again the sum is over the set T of equivalence classes of local group elements. Note that $\tilde{Q}$ is closed, so that it has a finite number of connected components (i.e. T is finite).

Now, let K be a simplicial decomposition of Q such that for each simplex $\sigma \in K$, the order of the isotropy group G_p of p is constant on the interior of σ (see [15]). Let $\tilde{K}$ be the simplicial decomposition of $\tilde{Q}$ induced by K, and for each $\sigma \in K$, denote by $\sigma_{(g)}$ the corresponding simplex in $\tilde{K}$ which lies in $\tilde{Q}_{(g)}$. As K is finite, let $\sigma^0, \sigma^1, \ldots, \sigma^k$ be an enumeration of the simplices in K so that $\tilde{K} = \{\sigma^i_{(g)} : 0 \leq i \leq k, (g) \in T\}$.

For each $\sigma^i_{(g)}$, let $p^i_{(g)}$ be a point on the interior of the simplex, and let $h^i_{(g)}$ be an element of $G_{p^i_{(g)}}$ such that $h^i_{(g)} \in (g)$. Note that the order $|(h^i_{(g)})_{G_{p^i_{(g)}}}|$ of the conjugacy class of $h^i_{(g)}$ in $G_{p^i_{(g)}}$, as well as the orders $|G_{p^i_{(g)}}|$ and $|C(h^i_{(g)})|$, are independent of the

choices of $p^i_{(g)}$ and $h^i_{(g)}$. Hence,

$$\begin{aligned}
\sum_{(g)\in T} \chi_{orb}(\tilde{Q}_{(g)}) &= \sum_{(g)\in T} \sum_{i=0}^{k} (-1)^{\dim \sigma^i_{(g)}} \frac{1}{|C(h^i_{(g)})|} \\
&\quad \text{(if there is no } \sigma^i_{(g)} \text{ for a specific } (g) \text{ and } i\text{, then let the term be zero)} \\
&= \sum_{(g)\in T} \sum_{i=0}^{k} (-1)^{\dim \sigma^i_{(g)}} \frac{|(h^i_{(g)})|}{|G_{p^i_{(g)}}|} \\
&\quad \text{as } |G_{p^i_{(g)}}| = |(h^i_{(g)})||C(h^i_{(g)})| \\
&= \sum_{i=0}^{k} \quad \sum_{(g)\in T:(g)\cap G_{p^i_{(g)}} \neq \emptyset} (-1)^{\dim \sigma^i_{(g)}} \frac{|(h^i_{(g)})|}{|G_{p^i_{(g)}}|} \\
&= \sum_{i=0}^{k} (-1)^{\dim \sigma^i_{(1)}} \frac{|G_{p^i_{(g)}}|}{|G_{p^i_{(g)}}|} \\
&= \sum_{i=0}^{k} (-1)^{\dim \sigma^i_{(1)}} \\
&= \chi(Q).
\end{aligned}$$

To finish the proof, suppose Q is almost complex, and note that for each $(g) \in T$, as $\tilde{Q}_{(g)}$ is on its own an orbifold, $E(\tilde{\Omega})$ is a representative of the Euler class of $\tilde{Q}_{(g)}$ in $H^*(\tilde{Q}_{(g)})$. Denote this class $e(g)$, and then $E(\tilde{\Omega})$ represents the element

$$\bigoplus_{(g)\in T} e(g) \in H^*_{orb}(Q),$$

which is an invariant of the connection. Note that if $\tilde{Q}_{(g)}$ has dimension $d_{(g)}$, then $e(g)$ is an element of $H^{d_{(g)}+2\iota(g)}_{orb}(Q)$.

Q.E.D.

Note that it is inessential that $\tilde{\Omega}$ be defined as being induced by a connection on Q; the theorem holds if we begin with an arbitrary connection on $\tilde{Q}$.

In the case of an almost complex, reduced orbifold Q of dimension n, the cohomology group $H^n_{orb}(Q)$ is isomorphic to the de Rham group $H^n(Q)$. Hence, the top part of $E(\tilde{\Omega})$ is a representative of the Euler class of Q with respect to this isomorphism. In the case that Q is not reduced, if i denotes the number of elements of T whose representatives act trivially, then $H^n_{orb}(Q)$ is isomorphic to $H^n(Q) \oplus H^n(Q) \oplus \cdots \oplus H^n(Q)$ (i copies). Then the top part of $E(\tilde{\Omega})$ is i copies of the Euler curvature form.

Definition 4.4.3 Let Q be an almost complex, closed, oriented orbifold, $E \to Q$ a vector bundle, and $\tilde{E} \to \tilde{Q}$ the induced bundle. The **orbifold Euler class** $e_{orb}(E)$ is the cohomology class represented by $E(\tilde{\Omega})$ in $H^*_{orb}(Q)$ for some connection $\tilde{\omega}$ on $\tilde{Q}$ with curvature $\tilde{\Omega}$.

Corollary 4.4.4 (The Second Poincaré-Hopf Theorem for Closed Orbifolds) Let X be a vector field on the closed orbifold Q with a finite number of zeros, and let $\tilde{X}$ be the induced vector field on $\tilde{Q}$. Then

$$\mathrm{ind}(\tilde{X}) = \chi(Q).$$

Proof:

This follows from the Poincaré-Hopf theorem for closed orbifolds [20], applied to each connected component of $\tilde{Q}$. Note that, as vector fields must be tangent to the singular set, a vector field with a finite number of zeros on Q will induce a vector field with a finite number of zeros on $\tilde{Q}$.

Q.E.D.

Again, it is inessential that we begin with a vector field on Q and pull back to $\tilde{Q}$.

4.4.1 Examples

Example 4.4.5 We start with the example of a single point $Q = \{p\}$ with the trivial action of a finite group G. In this case, the equivalence relation reduces to conjugation in the group. Then $\tilde{Q} = \{(p,(g)) : (g) \in T\}$, and the degree shifting number $\iota_{(g)} = 0$ for each $(g) \in T$ (see [3]).

The contribution of each connected component $\{(p,(g))\}$ of $\tilde{Q}$ to the orbifold cohomology is in $H^0_{orb}(Q)$, so that if n is the number of conjugacy classes in G,

$$H^d_{orb}(Q) = \begin{cases} \mathbb{R}^n, & d = 0, \\ \\ 0 & d \neq 0 \end{cases}$$

The curvature form of each point is the function $\Omega(p,(g)) = \frac{1}{|(g)|} = \chi_{orb}(p,(g))$, so that summing this value over each of the connected components of $\tilde{Q}$ gives

$$\begin{aligned} \sum_{(g)\in T} \frac{1}{|C(g)|} &= \sum_{(g)\in T} \frac{|(g)|}{|G|} \\ \\ &= 1 \\ \\ &= \chi(Q). \end{aligned}$$

Example 4.4.6 Let Q denote the $\mathbb{Z}_k$-teardrop, and then $\tilde{Q} = Q \sqcup \coprod_{i=1}^{k-1} \{(p,(i))\}$. The group $\mathbb{Z}_k$ acts trivially on each $(p,(i))$, so that the orbifold Euler characteristic of each

of these points is $\frac{1}{k}$. Then $\chi_{orb}(Q) = \frac{k+1}{k}$, so that

$$\begin{aligned}\int_{\tilde{Q}} E(\tilde{\Omega}) &= \chi_{orb}(Q) + \sum_{i=1}^{k-1} \chi_{orb}(p,(i)) \\ &= \tfrac{k+1}{k} + (k-1)\tfrac{1}{k} \\ &= 2 \\ &= \chi(Q).\end{aligned}$$

4.5 The Case With Boundary

We return to the case of an orbifold with boundary. A modification of the proof of 4.4.2 shows:

Theorem 4.5.1 (The Second Gauss-Bonnet Theorem for Orbifolds with Boundary) Let Q be a closed orbifold of dimension n with boundary M, and suppose Q carries a connection ω with curvature Ω. Let, $\tilde{\omega}$, $\tilde{\Omega}$, etc. be defined as in the previous section, and then

$$\int_{\tilde{Q}} E(\tilde{\Omega}) = \chi'(Q) - \frac{1}{2}\chi(M).$$

If Q is almost complex, then $E(\tilde{\Omega})$ represents an element of the cohomology ring $H^*_{orb}(Q)$ which is independent of the connection on Q.

Note that $\chi'(Q) = \chi(Q, M)$.

Proof:

Again, by the Gauss-Bonnet theorem for orbifolds with boundary,

$$\begin{aligned}\int_{\tilde{Q}} E(\tilde{\Omega}) &= \sum_{(g)\in T} \int_{\tilde{Q}_{(g)}} E(\tilde{\Omega}) \\ &= \sum_{(g)\in T} \chi'_{orb}(\tilde{Q}_{(g)}) - \tfrac{1}{2}\chi_{orb}(M_{(g)}).\end{aligned}$$

Using a simplicial decomposition of Q as above and the same counting argument, we have

$$\sum_{(g)\in T} \chi'_{orb}(\tilde{Q}_{(g)}) - \frac{1}{2}\chi_{orb}(M_{(g)}) = \chi'(\tilde{Q}) - \frac{1}{2}\chi(M).$$

Q.E.D.

Clearly, in the case where the dimension n of Q is even, this formula becomes

$$\int_{\tilde{Q}} E(\tilde{\Omega}) = \chi'(Q),$$

and in the case where n is odd,

$$\frac{1}{2}\chi(M) = \chi'(Q).$$

We now return to the proof of Proposition 3.2.4. Let $\tilde{\Psi}$ be defined in the natural way by taking the sum of Ψ on each connected component of $\tilde{Q}$. Then the relation $-d\tilde{\Psi} = \rho^* E(\tilde{\Omega})$ is immediate, as it is true on each connected component. Again, let $\tilde{X}$ denote the extension of the vector field X to $\tilde{Q}$. We modify the proof of Proposition 3.2.4 by applying in the first step Theorem 4.5.1 instead of the Theorem 3.2.2 as follows.

If the dimension $n = 2m$ of Q is even, then

$$\begin{aligned}
\chi'(Q) &= \int_Q E(\tilde{\Omega}) \\
&\quad \text{(by Theorem 4.5.1)} \\
&= \lim_{r\to 0^+} \int_{\tilde{Q}\setminus B_r(p)} \alpha^*\rho^*(E(\tilde{\Omega})) \\
&= -\lim_{r\to 0^+} \int_{\tilde{Q}\setminus B_r(p)} d\alpha^*(\tilde{\Psi}) \\
&= \lim_{r\to 0^+} \int_{\partial B_r(p)} \alpha^*(\tilde{\Psi}) - \int_{\tilde{M}} \alpha^*(\tilde{\Psi}) \\
&= \operatorname{ind}(\tilde{X}) - \int_{\tilde{M}} \alpha^*(\tilde{\Psi}),
\end{aligned}$$

and hence

$$\operatorname{ind}(\tilde{X}) = \chi'(Q) + \int_{\tilde{M}} \alpha^*(\tilde{\Psi}).$$

Note that the singular points p are taken to be those of $\tilde{X}$, and hence the $B_r(p)$ contains balls each singular point of $\tilde{X}$.

Making the identical modification to the proof in the case that $n = 2m+1$ is odd, we obtain

$$\chi'(Q) - \frac{1}{2}\chi(M) = \operatorname{ind}(\tilde{X}) - \int_{\tilde{M}} \alpha^*(\tilde{\Psi}),$$

and hence

$$\operatorname{ind}(\tilde{X}) = \chi'(Q) - \frac{1}{2}\chi(M) + \int_{M} \alpha^*(\tilde{\Psi}).$$

Now, note that in the case that Q admits a complex structure, as the cohomology class of Ψ is independent of the connection chosen, the cohomology class $\tilde{\Upsilon}$ of $\tilde{\Psi}$ in $H^*_{orb}(STQ_{|M})$ is similarly independent. In fact, it is clear that we can define $\tilde{\Upsilon}$ to be

the sum of the cohomology classes of the forms Ψ defined on each connected component of $\tilde{Q}$ from the connection, and then $\tilde{\Psi}$ would be a representative of the cohomology class $\tilde{\Upsilon}$. Hence, we have proven

Theorem 4.5.2 (The Second Poincaré-Hopf Theorem for Orbifolds with Boundary) Let Q be a compact oriented orbifold with boundary M, and suppose Q admits an almost complex structure. Let X be vector field on Q which has a finite number of singularities, all of which occuring on the interior of Q. Then with $\tilde{Q}$, $\tilde{X}$, etc. defined as above, we have

$$\mathrm{ind}(\tilde{X}) = \chi'(Q) + \tilde{\alpha}^* \tilde{\Upsilon}([\tilde{M}]).$$

Here, $\tilde{\alpha}^* \tilde{\Upsilon}([\tilde{M}])$ refers to the integral of any form representing the cohomology class $\tilde{\Upsilon}$ over the orbifold $\tilde{M}$; we have chosen to use this notation to emphasize the fact that the value of this integral is independent of the particular representative of $\tilde{\Upsilon}$ chosen.

Note that in the case that Q is a smooth manifold without boundary, both Theorem 3.4.2 and Corollary 4.5.2 reduce to the classical Poincaré-Hopf Theorem. Hence, both can be considered to be generalizations of this theorem to orbifolds with boundary, in the spirit of [22].

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Appendix A

A Note on Orbifold Euler Characteristics

This note is intended to clarify the various definitions of Euler Characteristics for that have been given for orbifolds and preclude any confusion which may arise. Throughout, let Q be a compact oriented orbifold and let X_Q denote its underlying topological space.

The first orbifold Euler characteristic was defined by Satake in [20], and was originally denoted $\chi_V(Q)$. This is the orbifold Euler characteristic used in this paper, and which we have denoted $\chi_{orb}(Q)$. It was defined as the index, in the orbifold sense, of a vector field on Q with isoloated singularities, or equivalently as the integral, in the orbifold sense, of the Euler curvature form $E(\Omega)$, defined in terms of an orbifold connection on Q. In general, it is a rational number. If $e(Q)$ denotes the cohomology class of $E(\Omega)$ and $e(X_Q)$ the usual Euler class of the underlying space of Q, then using the isomorphism between the de Rahm cohomology of Q as an orbifold and the singular cohomology of X_Q, it is easy to see that

$$\frac{\chi_{orb}(Q)}{\chi(X_Q)} e(Q) = e(X_Q).$$

In many ways, this Euler characteristic plays the role of the usual Euler characteristic of a manifold; as was discussed in Chapter 3, it can be defined in terms of suitable triangulation of the orbifold.

In the case that $Q = M/G$ is the quotient of a manifold by a finite group, another

orbifold Euler number was defined in [6] as

$$\chi(M,G) := \frac{1}{|G|}\sum_{g,h\in G: gh=hg} \chi(M^g \cap M^h),$$

where M^g denotes the fixed-point set of g (see also [12], where this is shown to be the Euler characteristic of the equivariant K-theory $K_G(M)$). In [16], this definition is generalized to the case of a general complex orbifold with groups acting as subgroups of $SL_n(\mathbb{C})$ as follows: If $e(p)$ denotes for each $p \in Q$ the number of conjugacy classes in the isotropy subgroup I_p, then define

$$\chi^o(Q) := \sum_{k=1}^{\infty} k\chi(e^{-1}(k)).$$

Note that the sum on the right has finitely many non-zero terms. Roan shows that this definition coincides with that of Dixon *et. al.* [6] in the case of a global quotient, and moreover that it does not coincide with the usual Euler characteristic of the underlying space of Q unless Q is nonsingular.

Similarly, in [1], the formula for $\chi(M,G)$ is genearlized to general orbifolds by defining

$$\chi_{orb}(Q) := \dim K^0_{orb}(Q)\otimes\mathbb{Q} - \dim K^1_{orb}(Q)\otimes\mathbb{Q}.$$

Here, $K_{orb}(Q)$ denotes the **orbifold K-theory** of Q, which is defined using complex orbifold vector bundles over Q and generalizes the equivariant K-theory of M/G.

We note that in the case of a reduced orbifold, these two invariants $\chi^o(Q)$ and $\chi_{orb}(Q)$ coincide. Indeed, using the notation of Chapter 4, for each $p \in Q$, $e(p)$ coincides with the number of distinct $(g) \in T$ such that $(p,(g)) \in \tilde{Q}$. Hence, for each positive integer k, the preimage $\pi^{-1}e^{-1}(k)$ of $e^{-1}(k)$ in $\tilde{Q}$ via the projection $\pi : \tilde{Q} \to Q$ is a

k-fold disjoint covering. Therefore, we have that

$$\begin{aligned}
\chi^o(Q) &= \sum_{k=1}^{\infty} k\chi(e^{-1}(k)) \\
&= \sum_{(g)\in T} \chi(\tilde{Q}_{(g)}) \\
&= \chi_{orb}(Q).
\end{aligned}$$

The last equality follows from the decomposition theorem for orbifold K-theory of Adem-Ruan ([1] Theorem 5.1, Corollary 5.6).

Appendix B

Relation to the Index Theorem for Orbifolds

Let Q be a compact orbifold that admits an almost complex structure, TQ its (orbifold) tangent bundle. Let E and F be two orbifold vector bundles over Q, and P an elliptic pseudodifferential operator $C^\infty(E) \to C^\infty(F)$ from the smooth sections of E to those of F. Then if $\rho : T^*Q \to Q$ denotes the projection of the tangent bundle, the symbol σ of P gives an isomorphism $\sigma : \rho^*E \to \rho^*F$ off of the zero-section of T^*Q, and hence an element of $K_{orb}(T^*Q)$.

Let $u \in K_{orb}(T^*Q)$ denote such a class in the orbifold K-theory of $T^*Q \cong TQ$. Let $\tilde{Q}, \tilde{Q}_{(g)}$, etc. be defined as in Chapter 4, and let $\tilde{E}, \tilde{F}$ denote the induced bundles on $\tilde{Q}$. Let $\tilde{E}_{(g)}$ denote the induced bundle restricted to the connected component $\tilde{Q}_{(g)}$ of $\tilde{Q}$, and let $u_{(g)}$ denote the class of $\tilde{E}_{(g)} - \tilde{F}_{(g)}$ in $K_{orb}(\widetilde{TQ_{(g)}}) = K_{orb}(T\tilde{Q}_{(g)})$ with the symbol given by the restriction of P. With this setup, Kawasaki [10] showed that the orbifold index

$$\mathrm{ind}_{orb}(P) := \dim[\ker(P)] - \dim[\mathrm{coker}(P)]$$

is given by

$$\mathrm{ind}_{orb}(u) = (-1)^{\dim Q} \langle \mathrm{ch}(u) \cup \mathcal{T}(Q), [TQ] \rangle + \sum_{(1) \neq (g) \in T} \frac{(-1)^{\dim \tilde{Q}_{(g)}}}{|C(g)|} \langle \mathrm{ch}(u_{(g)}) \cup \mathcal{T}(\tilde{Q}_{(g)}), [T\tilde{Q}_{(g)}] \rangle$$

(see also [7]). Here, $\mathrm{ch}(u)$ denotes the Chern character of u ($\mathrm{ch}(E) - \mathrm{ch}(F)$), and $\mathcal{T}$ the Todd class of te (complexified) tangent bundle.

We we may define the **orbifold Chern character** $\mathrm{ch}_{orb}(E)$ of a complex vector

bundle E and the **orbifold Todd class** $\mathcal{T}_{orb}(Q)$ in an analogous manner to the orbifold Euler class: to a bundle E with connection ω we associate the induced bundle $\tilde{E}$ over $\tilde{Q}$ with connection $\tilde{\omega}$, and then construct the the Chern character and Todd class from the curvature of this connection in the usual manner (the bundle is taken to be the tangent bundle in the case of the Todd class). It is clear that these forms represent elements of $H^*(Q)$ that are independent of the connection, and that they can be given as a sum of ordinary characteristic classes of the $\tilde{Q}_{(g)}$. In other words, we have

$$\mathrm{ch}_{orb}(\tilde{E} - \tilde{F}) = \sum_{(g)\in T} \mathrm{ch}(\tilde{E}_{(g)} - \tilde{F}_{(g)})$$

and

$$\mathcal{T}_{orb}(Q) = \sum_{(g)\in T} \mathcal{T}(\tilde{Q}_{(g)}).$$

We note that the factors $\frac{1}{|C(g)|}$ appear in the above formula because Kawasaki worked with a definition of orbifold which required that it was reduced. Hence, his $\tilde{Q}_{(g)}$ correspond to our $(\tilde{Q}_{(g)})_{red}$, and the corresponding factors are built into our definition of the integral.

With this, by reversing the orientation of the odd-dimensional components of $\tilde{Q}$, Kawasaki's index formula can be written as

$$\mathrm{ind}_{orb}(u) = \langle \mathrm{ch}_{orb}(u) \cup \mathcal{T}_{orb}(Q), [\widetilde{TQ}] \rangle.$$

We note that the cup product here is **not** the orbifold cup product defined in [3], but rather the pointwise product induced by the wedge product. In particular, forms on $\tilde{Q}_{(g)}$ cupped with forms on $\tilde{Q}_{(h)}$ yield 0 when $(g) \not\sim (h)$.